ASP-110 - First Day/Continued Fractions

*Administer Quiz

Let's start with a short 10 minute quiz. Do your best, do not be concerned if you've forgotten how to do some of the problems.

*Take Attendance

I'm glad to see all of you here on time.

My name is Mitchell Schoenbrun. I am unconcerned as to how you address me. Other professors you meet at USF might be more particular.

I've been teaching mathematics for about 15 years. I also have a career as a computer programming consultant. During my office hours you are welcome to ask me about computer programming as a career.

This class will only meet 6 times, so it is important that come to every class, and it is important that you come on time. If you have difficulty getting places on time, consider aiming for 10 minutes early each day. There is no harm in being early.

I maintain a website for the class at http://schoenbrun.com/usf. You may find information there that is useful.

Please take a close look at the syllabus for the course. For each course you take here at USF a syllabus will be available. It is always a good idea to look it over in detail. Last semester I had a student who claimed half way through that he did not know that he needed a book for the class. Often the dates for mid-term exams will be listed.

As I said previously I expect everyone to come to class each day on time. To get an excused absence you must provide me with a note from your advisor indicating that they are convinced that your absence was unavoidable.

When in class I expect you to turn off or mute your cell phones. The only possible use for a laptop or computer pad will be if you use it to take notes.

Some words about the curriculum.

In this class we will review some mathematics that you will need for Calculus. This is not the main purpose of this class however it will be useful for you to take this opportunity to refresh yourselves on mathematics that will be needed.

Some of this mathematics will be familiar to some of you, but there will be new mathematics that none of you have seen before. Learning this mathematics is not the main purpose of this class.

You will be given homework assignments in this class. You will need to spend an adequate amount of time and effort on this homework. It will not be graded on the basis of answering questions perfectly. It will be graded based on whether you show that you have made a serious and sincere effort to do the problems.

There may be some short quizzes during class. There is no final exam for the class, however there will be a written assignment that you will be asked to present on the last day of class.

This brings me to the actual purpose of this class. Most if not all of you will find this first semester at USF academically difficult. This is completely normal. Most college students find that their first semester is the hardest. Most of you will have to adjust your behavior. That is you will have to work harder than you did in high school. You will have to spend less time watching movies and TV. You will have to spend less time socializing. You will have to spend less time using social media apps. This will be required for your success and survival here. I do not say this as a warning. I say this out of personal experience.

The purpose then is to give you a taste of what will be expected of you during the semester. To some degree your grade will reflect whether you have figured this out.

Are there any questions?

I assume most of you are familiar with this picture.

The dimension of this picture are 30" x 21".

That is a ratio of about 1.43.

The frame around the head has a similar ratio, about 1.45.

Both of these ratios are reasonably close to the mathematical value $\frac{1+\sqrt{5}}{2} \approx 1.61$.

This ratio is called the golden mean or the golden section, or the golden number.

This ratio appears quite frequently in Art, Architecture, and quite curiously in nature.

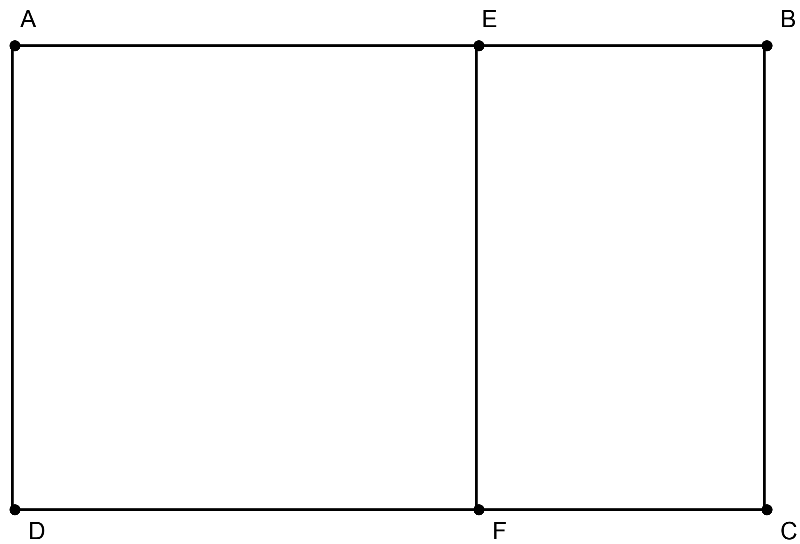
We are going to take a look at this number through the subject of **Continued Fractions**.

First I'd like to show you a geometric interpretation of the ratio.

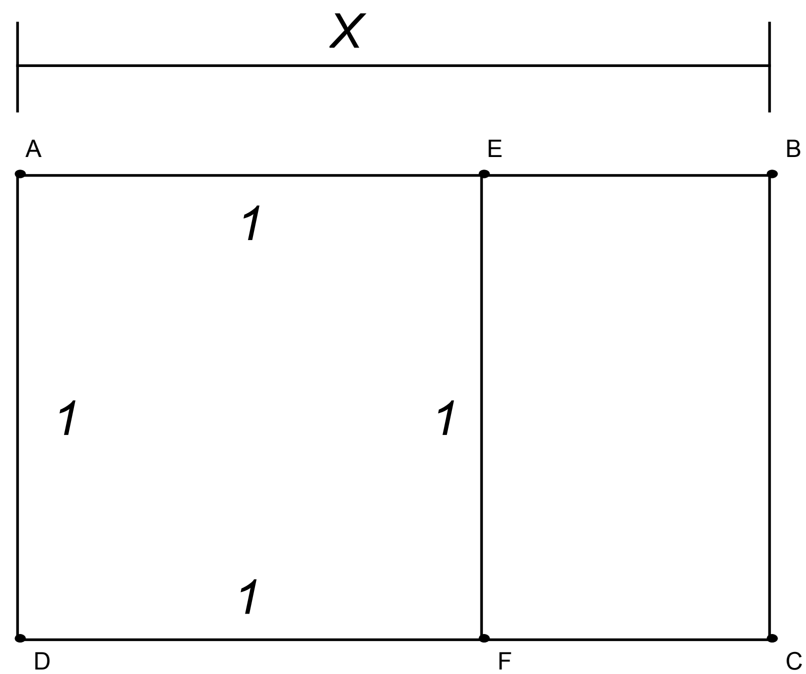
Assume that you have a rectangle with the following properties.

If you break the rectangle up into a square and a rectangle, let the remaining rectangle be similar to the original rectangle.

* What does it mean for two rectangles to be similar?
* What does it mean for any two geometric objects to be similar?


A
E
B
D
F
C

If we set the length of the square to be 1, and the measure of line segment $m\overline{AB} = x$



We see because of similarity the following relationships holds.

$$\frac{m\overline{AB}}{m\overline{BC}} = \frac{m\overline{BC}}{m\overline{BE}}$$

This means that

$$\frac{x}{1} = \frac{1}{x-1}$$

cross multiplying we find that

$$x^2 - x = 1$$
$$or$$
$$x^2 - x - 1 = 0$$

* This is what type of Equation?  (A quadratic)
* What are the methods we can use to solve this?  (Factoring, Completing the Square, Quadratic Formula)

* What is the Quadratic formula?

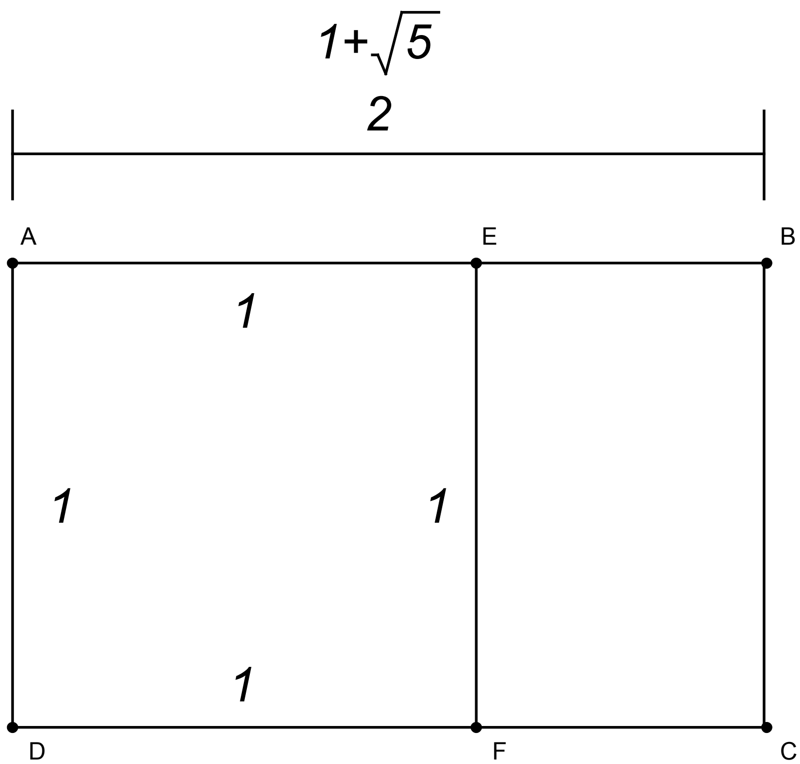
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

What then is the solution of the equation?

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

So in our diagram we have




This number is the  golden ratio.

**Continued Fractions**

We continue our investigation of the golden ratio by taking a digression.
There are a number of ways to represent a real number.
Examples are:

$7 \quad \frac{3}{2} \quad 6.2 \quad .33\bar{3} \quad 52_8 \quad \sqrt{7} \quad \pi$

I'm going to introduce a new way to represent a real number that has the following form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

This is called a **continued fraction**.

Note that the numerators of the fractions are always 1. An example would be:

$$2 + \cfrac{1}{3 + \cfrac{1}{4}}$$

Any rational number can be converted to a continued fraction

Example:

$$\frac{23}{14} = 1 + \frac{9}{14} = 1 + \cfrac{1}{\cfrac{14}{9}} = 1 + \cfrac{1}{1 + \cfrac{5}{9}} = 1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{9}{5}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{5}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{5}{4}}}} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4}}}}$$

An alternate and more compact notation is:

$$a_0 + \frac{1}{a_1 +} \frac{1}{a_2 +} \frac{1}{a_3} \dots$$

Example: $1 + \frac{1}{1 +} \frac{1}{1 +} \frac{1}{1 +} \frac{1}{4}$

It is straight forward to collapse a continued fraction back to it's fractional form:

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\frac{5}{4}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\frac{4}{5}}}=1+\cfrac{1}{1+\cfrac{1}{\frac{9}{5}}}=1+\cfrac{1}{1+\frac{5}{9}}=1+\cfrac{1}{\frac{14}{9}}=1+\frac{9}{14}=\frac{23}{14}$$

Given a rational number in continued fraction form, it is useful to look the value of partial versions

$$[1]+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4}=1.0$$

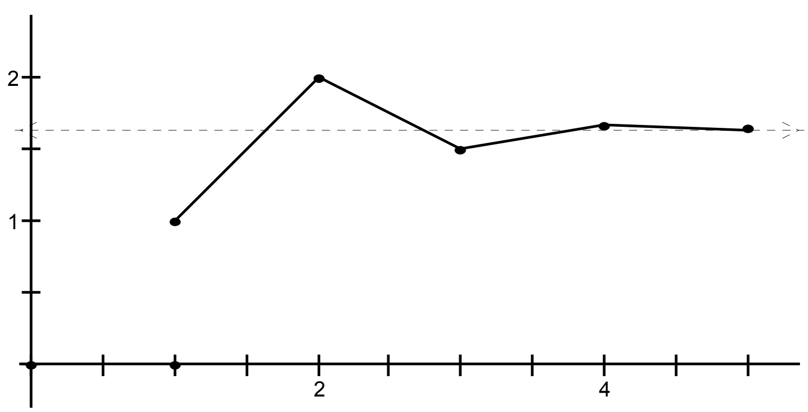
$$1+\frac{1}{1}=2.0$$

$$1+\frac{1}{1+}\,\frac{1}{1}=\frac{3}{2}=1.5$$

$$1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1}=\frac{5}{3}=1.\overline{6}$$

$$1+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4}=\frac{23}{14}=1.6\overline{428571}$$

Graphing these numbers we by the number of terms we see a very clear pattern




Each successive partial is closer to the original value, they appear to oscillate above and below the value.

This turns out to be the case, and we will find this quite useful.

**The continued fraction of an irrational number**

Let's start with a simple quadratic equation.

$$x^2 - 3x - 1 = 0$$

To find the positive solution of this equations you can use the quadratic formula, finding:

$$x = \frac{3 \pm \sqrt{9+4}}{2} = \frac{3+\sqrt{13}}{2} \approx 3.302775638$$

If instead you take a more naive approach you might do the following

$$x^2 - 3x - 1 = 0$$
$$x^2 = 3x + 1 \quad \text{Dividing by } x$$
$$x = 3 + \frac{1}{x}$$

This might seem pointless as you still have the variable $x$ on both sides of the equation. But note that you have an expression for $x$ that can be plugged into the right side.

$$x = 3 + \cfrac{1}{3 + \cfrac{1}{x}}$$

This can be repeated indefinitely

$$x = 3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{\ddots}}}}$$

Well this is a little startling. While you must be aware that the decimal expansion of the solution cannot repeat because $\sqrt{13}$ is an irrational number.
But it appears that the continued fraction expansion, while infinite, is quite regular:

$$3 + \frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3}\cdots$$

Recalling our previous calculation of the truncated values, we now would seem to have a way to approximate our solution with a rational number to any degree of accuracy we want. Let's check this out.

$$x = \frac{3+\sqrt{13}}{2} \approx 3.302775638$$
$$x_0 = 3.0$$
$$x_1 = 3 + \frac{1}{3} = 3.\overline{3}$$
$$x_2 = 3 + \frac{1}{3+}\frac{1}{3} = \frac{33}{10} = 3.3$$
$$x_3 = 3 + \frac{1}{3+}\frac{1}{3+}\frac{1}{3} = \frac{109}{33} = 3.\overline{30}$$
$$x_0 = 3 + \frac{1}{3+}\frac{1}{3+}\frac{1}{3+}\frac{1}{3} = \frac{360}{109} \approx 3.302752294$$

So with 5 levels of the continued fraction expansion we have a number correct to 6 digits. In addition we have this approximation as a fraction with a 3 digit numerator and denominator.

This brings us back to the question of the Golden Mean

$$x^2 - x - 1 = 0$$
$$x^2 = x + 1$$
$$x = 1 + \frac{1}{x}$$

So the expansion of

$$\frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$$

This is a surprisingly simple expansion of this somewhat curious number.   Looking at the partial approximations will be even more curious:

$$x_n = \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}$$

Does anyone recognize the sequence in the numerator and the denominator?

**The Fibonacci Sequence**

The Fibonacci sequence is named after an Italian mathematician Leonardo Fibonaci who lived around 1200 in Pisa Italy.

The sequence is very easily generated.

You start with the numbers 1 and 1.  Each following number is the sum of the previous two numbers.   Sometimes this is written this way.

$$a_0 = a_1 = 1$$
$$a_n = a_{n-1} + a_{n-2}$$

This is known as a recursive definition.

This sequence turns out to appear commonly in nature.

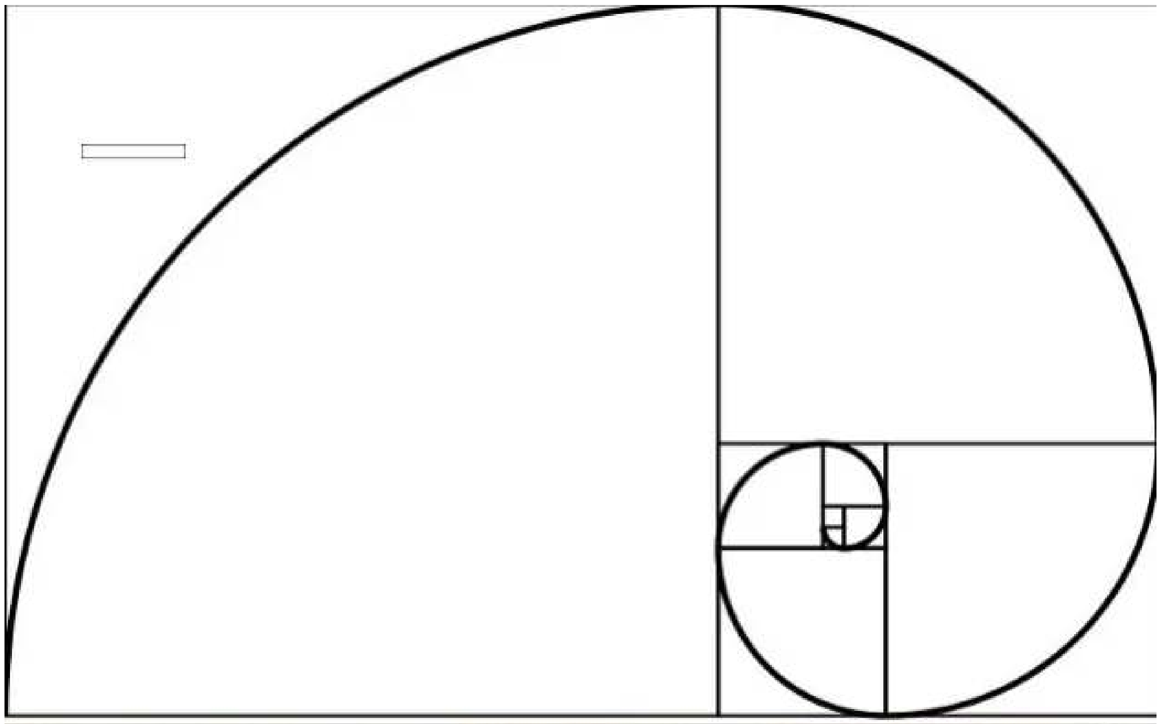
Some examples:

The number of petals in a flower commonly follows the Fibonacci sequence.  Examples are

Lilies with 3 petals
Buttercups 5 petals
Chicory  21 petals
A daisy with 34.

Pine cone patterns

Sea Shell Chambers, which also follow a curve called logarithmic spiral.



The Arms of spiral Galaxies also follow this spiral.

As do Hurricane patterns.