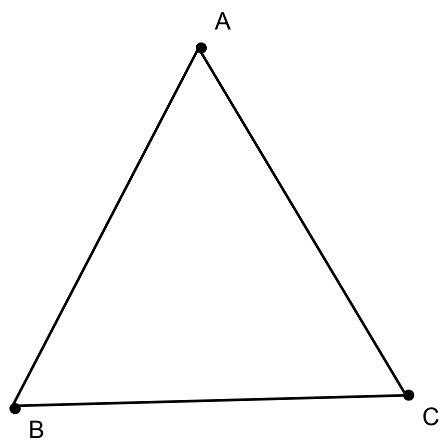
ASP-110 - Geometry

*Take Attendance
*Collect Homework

We're going to take a look at geometry, in particular some features of triangles.

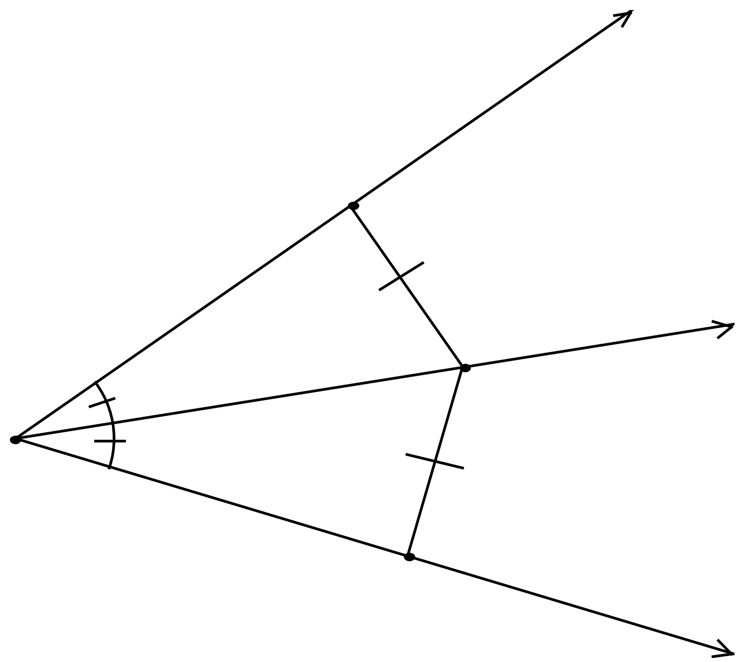
Let's start with a general triangle ABC



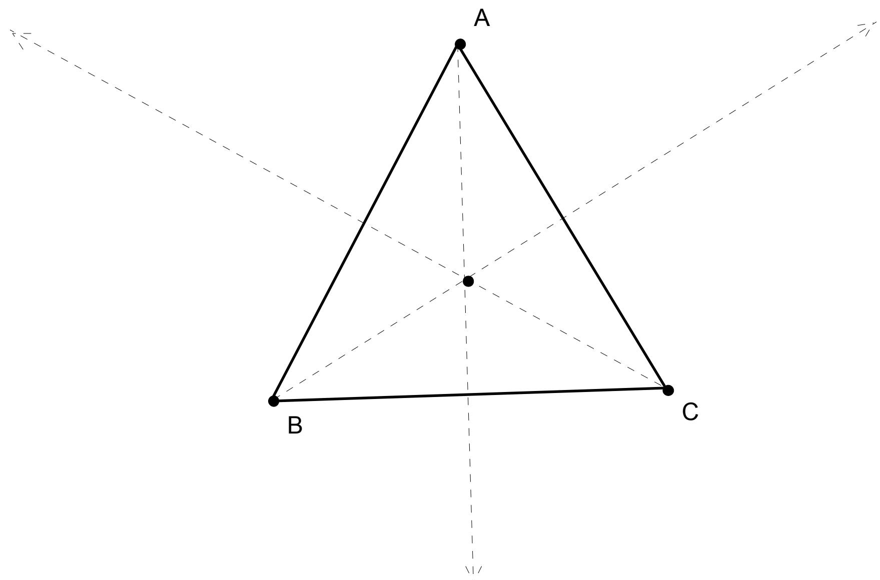
with a circle circumscribed around it.

You may recall that 3 points determine a circle, so there is only one such circle.

To find this circle, recall that every point on a angle bisector is equidistant from the rays of the original angle.

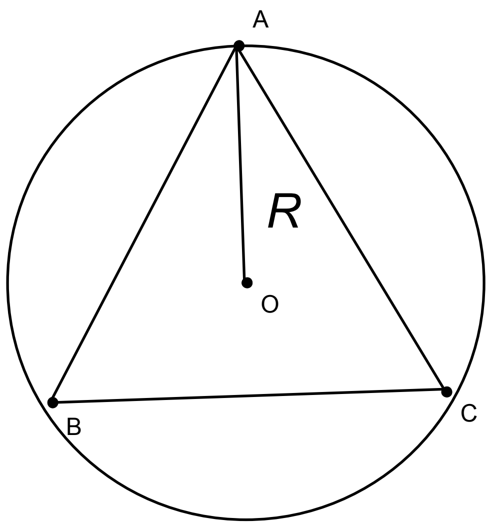


So if we construct the 3 angle bisectors of a triangle, and they meet at a point, that point will be equidistant from the vertices of the triangle. We will prove later that these bisectors must intersect at a single point.



So this point O will be the center of the circumscribing circle.

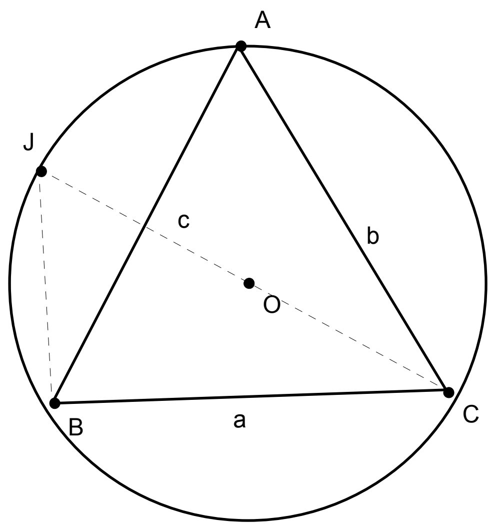
Call the radius of this circle *R*.



Next we draw a diameter through the center point O and C and label its 2nd intersection with the circle point *J*.

Also draw the line *JB* creating right triangle *JBC*.

How do we know this is a right triangle????
(It's inscribed in a semi-circle.)



Note that $\sin J = \frac{a}{CJ} = \frac{a}{2R}$

Now note that $\angle J = \angle A$ Why?
(They are both inscribed in the same arc)

So we have $\sin A = \frac{a}{2R}$

We can repeat this process with angles B and C finding that

$\sin B = \frac{b}{2R}$ and $\sin C = \frac{c}{2R}$

Re-arranging we find that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Who can recall the name of this theorem from Trigonometry?

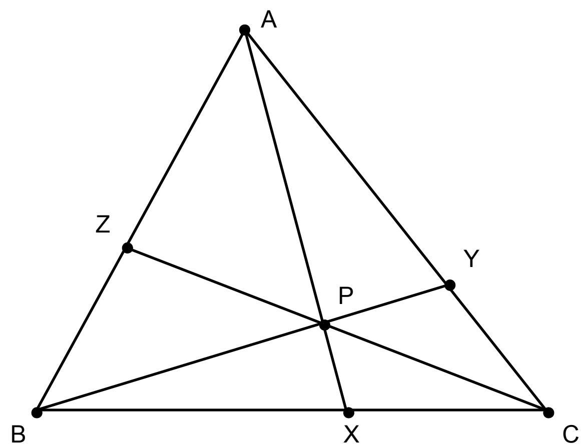
(The Law of Sines)

Note that we have the additional information that $\frac{x}{\sin x} = 2R$ where R is the radius of a circumscribing circle. This will come in handy later.

Ceva's Theorem

The line segment joing a vertex of a triangle to any point on the opposite side is called a **cevian**.

So in the following triangle:



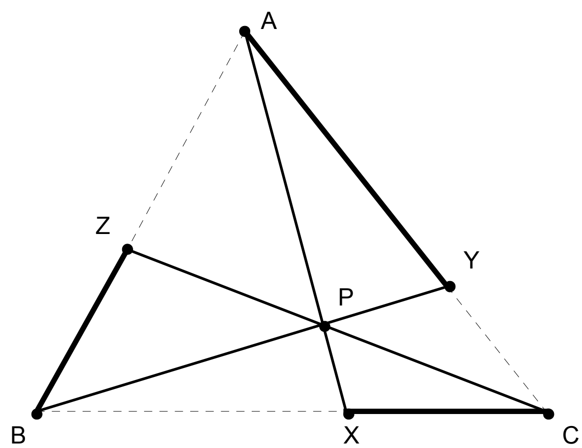
The line segments *AX*, *BY* and *CZ* are cevians.

This term comes from the name of the Italian mathematician Giovanni Ceva who published the following theorem in 1678.

Theorem: If the three cevians AX, BY and CZ are concurrent (they meet at a point P) then

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1$$

That is the product of the lengths of the alternating segments below are equal.



Proof:

Recall that the areas of triangles with equal altitudes are proportional to the lengths of the bases. From our diagram

$$\frac{BX}{XC}=\frac{(ABX)}{(AXC)}=\frac{(PBX)}{(PXC)}$$

Using the (ABC) notation to mean the area of triangle ABC

But if to fractions with different numerator and denominators are the same, a fraction whose numerator is the differences of the numerators and whose denominator is the difference of the denominators has the same value.

Eg: $\frac{4}{8}=\frac{1}{2}=\frac{4-1}{8-2}=\frac{3}{6}$

So $\frac{BX}{XC}=\frac{(ABX)}{(AXC)}=\frac{(PBX)}{(PXC)}=\frac{(ABX)-(PBX)}{(AXC)-(PXC)}=\frac{(ABP)}{(CAP)}$

Similarly

$\frac{CY}{YA}=\frac{(BCP)}{(ABP)}$ and $\frac{AZ}{ZB}=\frac{(CAP)}{(BCP)}$

If we multiply these three expressions we find

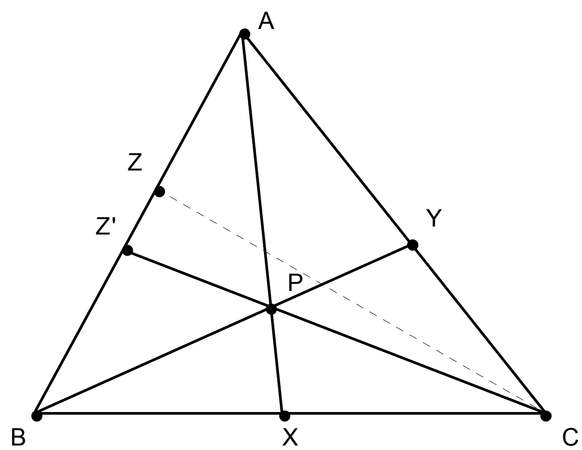
$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=\frac{(ABP)}{(CAP)}\cdot\frac{(BCP)}{(ABP)}\cdot\frac{(CAP)}{(BCP)}=1$$

The converse of this theorem (What is a converse) also holds.

If $\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1$ then three cevians meet at a point.

Proof: Assume that AX and BY meet at point P.

Let the point that where the third cevian through P meets AB be Z'.



By the theorem we just proved,

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ'}{Z'B}=1$$

using our assumption we find that

$$\frac{AZ}{ZB}=\frac{AZ'}{Z'B}$$

But this can only be true of Z and Z' are the same point.