ASP-110 - Limits

*Take Attendance
*Collect Homework

Today we will talk about a subject that will come up in your Calculus course, limits. We will cover limits in a way that was traditional for a Calculus course a few decades ago, but no longer is covered until a later course in Analysis.
This subject is conceptually hard, however the mathematics requires is limited to things you learned in your Algebra courses.

The idea of a limit

$$\lim_{x \to c} f(x) = L$$

We might read this as,

As x approaches c, $f(x)$ approaches L.

or

for x close to c, $f(x)$ is close to L

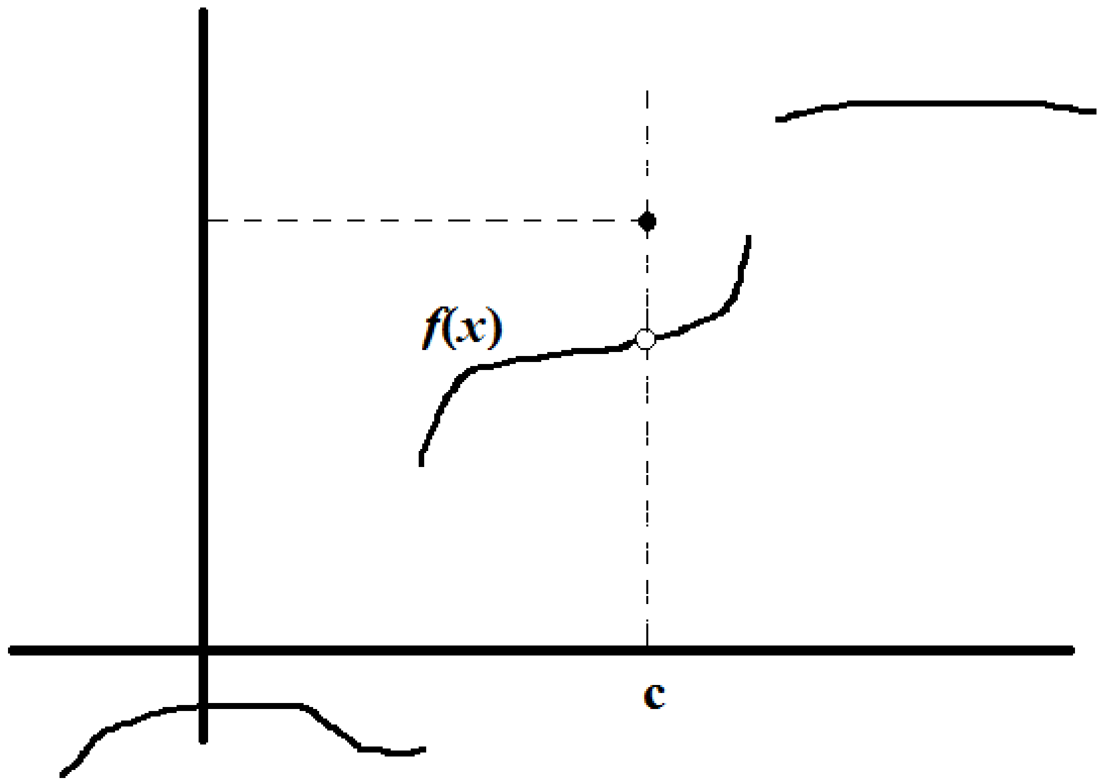
or even

if x is approximately equal to c, then $f(x)$ is approximately equal to L.

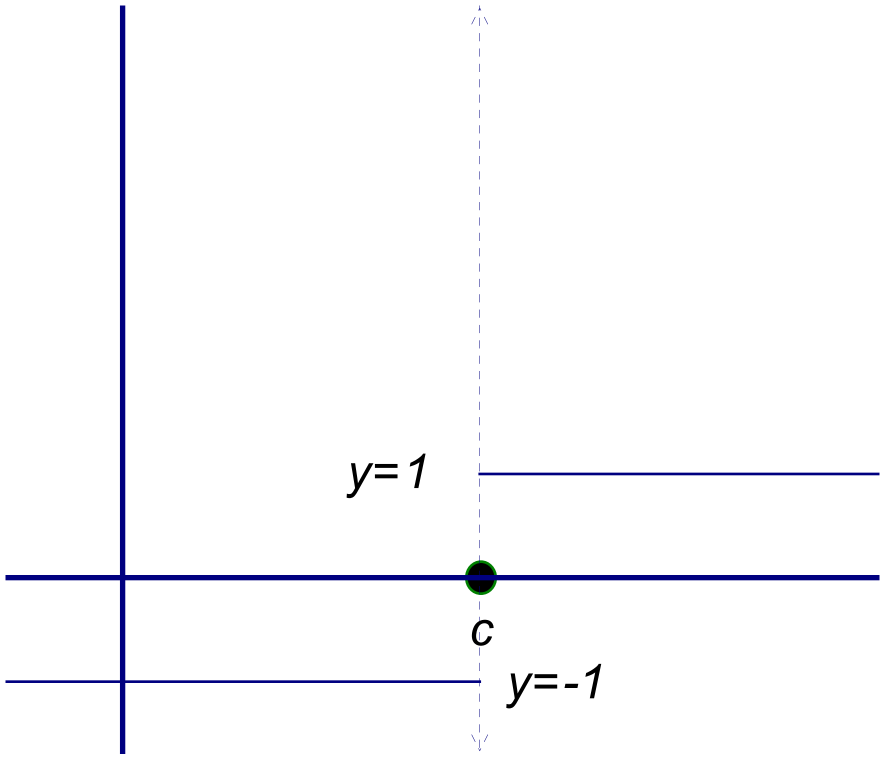
Notice that nothing is said about the value of $f(c)$.

It does not matter whether $f(x)$ is defined at c and if so, how it is defined there.

The only thing that matters is how f is defined near c.



Notice in this diagram that $f(x)$ is a broken curve and has a peculiar value at c, but it still has a limit at c.



For this function

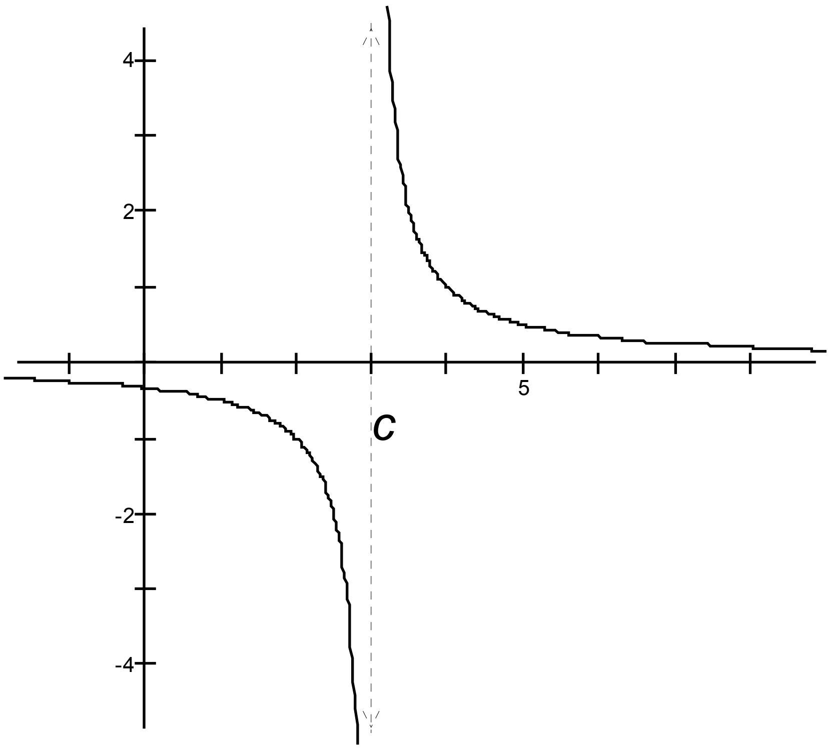
$$f(x)=\begin{cases}1, x>c\\0, x=c\\-1, x<c\end{cases}$$

we see that as $f(x)$ approaches -1 as x approaches c from the left, but $f(x)$ approaches 1 as x approaches c from the right.

So $\lim_{x\to c} f(x)$ does not exist.

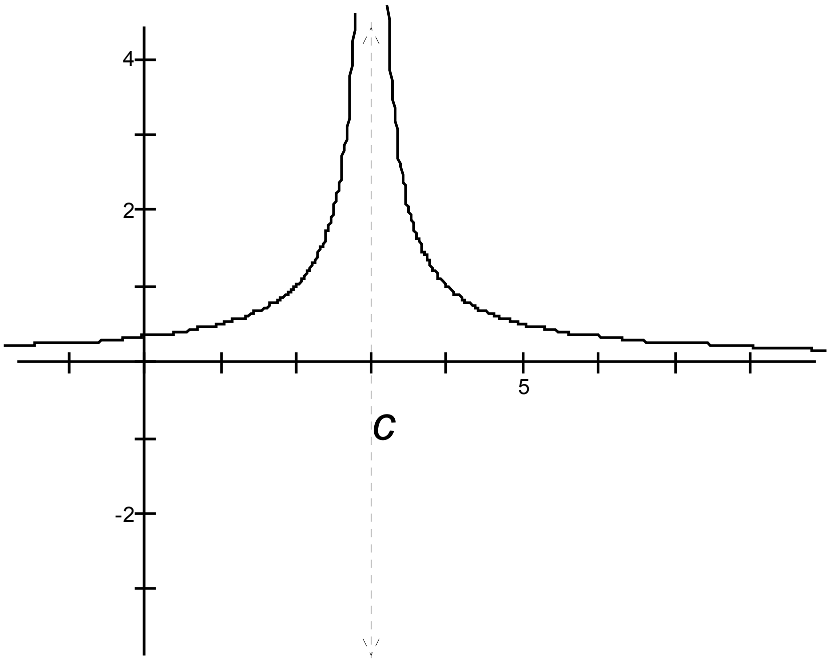
Some other examples in which a limit does not exists are:

$$f(x) = \frac{1}{x-c}$$



As one approaches from the left $f(x)$ decreases endlessly, while as one approaches from the right $f(x)$ increases endlessly.

$$f\left(x\right)=\frac{1}{\left|x-c\right|}$$



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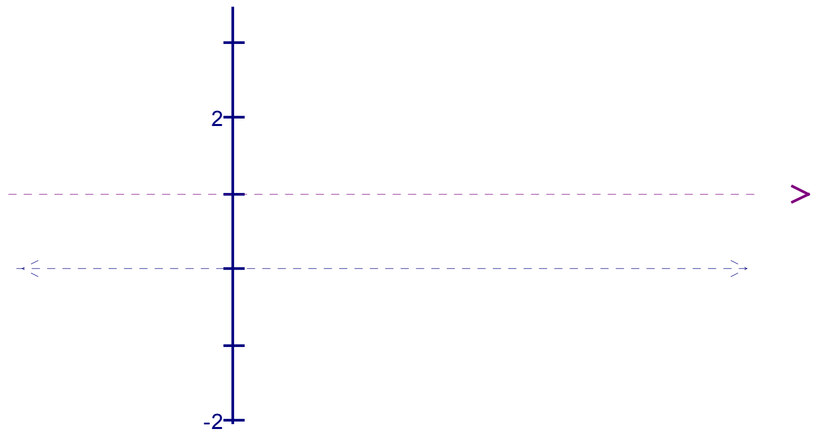
Here $f(x)$ increases from both directions, however it does not approach a finite value.



This function

$$f(x)=\begin{cases}1, & x \text{ rational}\\ 0, & x \text{ irrational}\end{cases}$$

Can't have a limit anywhere because it cannot stay close to either 0 or 1 as you approach any number.

Definition of a limit

The previous description was to give you a feeling for what a limit is.

Here we are going to provide here a precise mathematical definition of what

$$\lim_{x \to c} f(x) = L$$

means.

That is to say, we can make $f(x)$ as close to L by requiring that x is sufficiently close to c.

First think of a very small number and call it ε, the Greek letter epsilon. In mathematics this letter is usually used to indicate a very small number.

If $\lim_{x \to c} f(x) = L$ then we can be sure that $\left|f(x) - L\right| < \varepsilon$

if x is close enough to c.

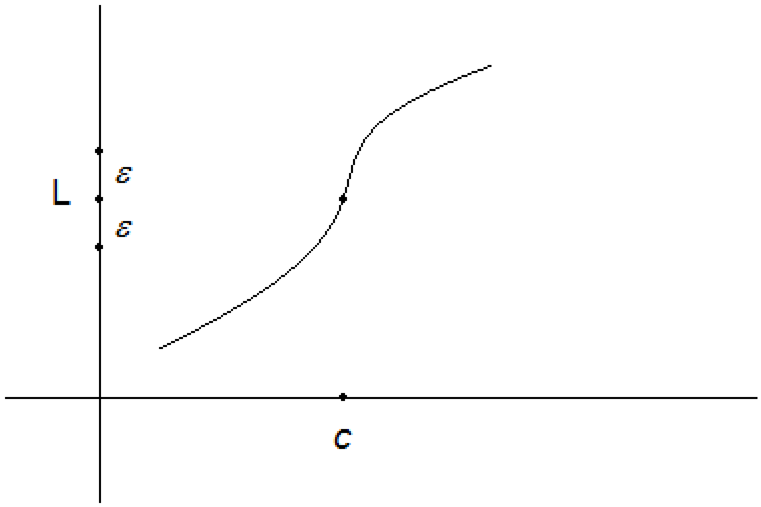
How close does it need to be? Well there must be some number δ, the Greek letter delta where as

$0 < \left|x - c\right| < \delta$ implies that $\left|f(x) - L\right| < \varepsilon$

So here is the formal definition of a limit

$$\lim_{x \to c} f(x) = L \text{ iff } \begin{cases} \text{for each } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ \text{if } 0 < \left|x - c\right| < \delta \text{ then } \left|f(x) - L\right| < \varepsilon \end{cases}$$

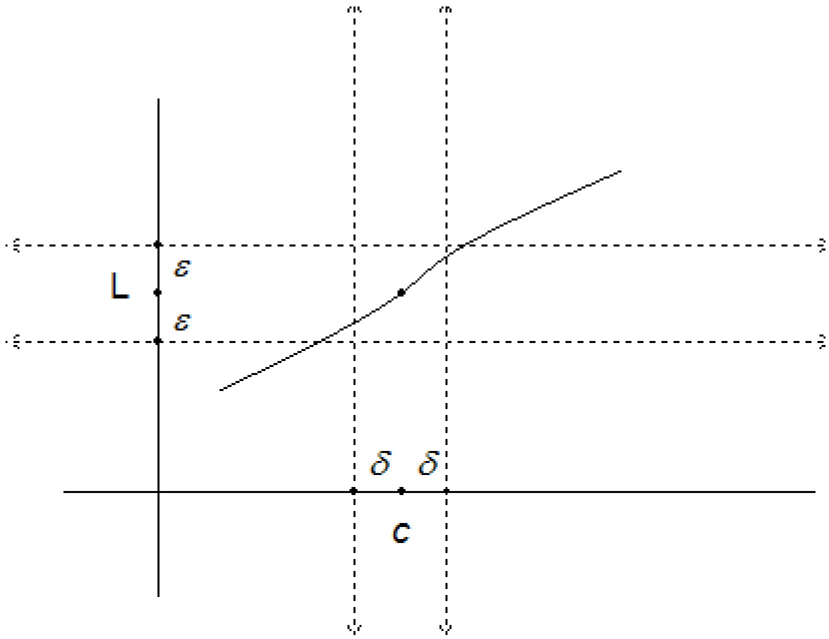
This definition takes a little getting used to.



We have some function *f*(*x*) whose value we think is *L* when close to *x*=*c*.

We choose an interval on the *Y*-axis around *L* that is within ε of *L*.

We want to find some δ so that the interval around *c* within δ of *c* guarantees that we are within ε of *L*.



If we can do this for every ε then we have proven that *L* is the limit at *c*.

Note: that for *L* to be the limit, it does not matter if *f* is defined at *c* nor does it matter what's value is at *c*.

Note: that the choice of δ can depend on ε.   There does not have to be a single δ that works for all ε.

Doing a epsilon delta proof can seem confusing at first. Keep in mind the hard part is finding the right delta. Usually you have to work backwards to find the right one.

Also note that if you find a δ that does work, then you all fractions of that number will also work, since they constrain the value of the function even more.

Example:

Prove that $\lim_{x \to 1} x = 1$

Proof:

Given that $\varepsilon > 0$ we must find a $\delta > 0$ such that

$|x-1| < \delta$ implies that $|f(x)-1| = |x-1| < \varepsilon$

This is really simple.

We choose $\delta = \varepsilon$

Which means that

$|x-1| < \delta = \varepsilon$ implies that $|x-1| < \varepsilon$

So we have shown that by our choice of δ

$|f(x)-1| < \varepsilon$

This proves that 1 is the limit.

That was a particularly simple example.

Note that we could substitute any constant value *c* for 1 and the proof would be identical.

Example:

Prove that $\lim_{x\to 2}(2x-1)=3$

Recall that because $2\cdot 2-1=3$ this does not prove the limit is 3 as the value of *f* at 2 is irrelevant.

Proof:

Given that $\varepsilon > 0$ we must find a $\delta > 0$ such that

$|x-c|=|x-2|<\delta$ implies that $|f(x)-L|=|(2x-1)-3|<\varepsilon$

We start by working backwards.

$$|(2x-1)-3|=|2x-4|=2|x-2|$$

Since we want $|x-2|<\delta$
$2|x-2|<2\delta$

So we can choose $\delta=\dfrac{\varepsilon}{2}$

Which means that

$$|(2x-1)-3|=2|x-2|<2\frac{\varepsilon}{2}=\varepsilon$$

So we have shown that by our choice of δ

$$|(2x-1)-3|<\varepsilon$$

This proves that 3 is the limit.

Example:

Prove that $\lim_{x \to c} ax = ac$ where a and c are arbitrary constants.

Proof:

Given that for $\varepsilon > 0$ we must find a $\delta > 0$ such that

$|x - c| < \delta$ implies that $|f(x) - L| = |ax - ac| < \varepsilon$

We choose $\delta = \dfrac{\varepsilon}{|a|}$

So

$$|x - c| < \frac{\varepsilon}{|a|}$$
$$|a||x - c| < \varepsilon$$
$$|ax - ac| < \varepsilon$$

So the limit is proven

Example:

Prove that $\lim_{x \to 4} \sqrt{x} = 2$

Proof:

Given that for $\varepsilon > 0$ we must find a $\delta > 0$ such that

$|x-4| < \delta$ implies that $|f(x) - L| = |\sqrt{x} - 2| < \varepsilon$

We choose $\delta < 4$ and $\delta = 2\varepsilon$, or $\varepsilon = \frac{\delta}{2}$

So we have $0 < |x-4| < \delta$

Since $\delta < 4$ it must be the case that $x > 0$ for if $x \le 0$ then $\delta > |x-4| \ge 4$

Note that: $|x-4| = \left|(\sqrt{x}+2)(\sqrt{x}-2)\right| |\sqrt{x}+2||\sqrt{x}-2| < \delta$

But since $x > 0$ then we must have $2 < |\sqrt{x}+2|$

So $|\sqrt{x}+2||\sqrt{x}-2| < \delta$ implies $2|\sqrt{x}-2| < \delta$ or

$$|\sqrt{x}-2| < \frac{\delta}{2} = \varepsilon$$

So $|\sqrt{x}-2| < \varepsilon$ proving that 2 is the limit.

In class, please try the following:

Prove that $\lim_{x \to 4} (x-3) = 1$