The Infinitude of Primes
Proofs, indirect and direct

I'm going to start with a simple indirect proof that you are no doubt familiar with. This is an indirect proof, or proof by contradiction that there are an infinite number of primes. This proof is attributed to Euclid.

We start with the assumption of the negation, that there is a finite number of primes.

We label these primes $p_1, p_2, p_3, \cdots, p_n$

We construct a number $P = p_1 \cdot p_2 \cdot p_3 \cdot \ldots p_n + 1$

Either P is a prime or it is composite and therefore has a prime divisor q.

But clearly P is not a member of the set $\{p_1, p_2, p_3, \cdots, p_n\}$ nor can q be a member because every p_i has a remainder of 1 when dividing P.

This contradiction shows our assumption is wrong so there are an infinite number of primes.

This proof has a number of nice features. It is straight forward and simple to understand. Also, given any finite set of primes, it gives us a way to construct a larger prime.

It is however a proof by contradiction. There's nothing logically wrong with a proof by contradiction, but in general direct proofs are preferred. Why is this? Well often direct proofs will give us more insight into the theorem, whereas indirect proofs often only show its correctness.

That said, I'd like to describe a direct proof of the infinitude of primes.

First we start with the well know idea that the harmonic series

$$H=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n}$$ is divergent.

By this I mean that given any large number N there is an i such that $\sum_{n=1}^{i}\frac{1}{n}>N$.

This is easy to show as follows. Group the series as follows:

$$H=\frac{1}{1}+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\cdots+\frac{1}{16}\right)\cdots$$

where the number of terms doubles in each group. Now compare this with the series

$$G=\frac{1}{1}+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\cdots+\frac{1}{16}\right)\cdots$$

But simplifying G we see that

$$G=\frac{1}{1}+\frac{1}{2}+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)\cdots$$

Which is clearly divergent.

Since each term in the harmonic series is greater than each term in this series, it must be the case that $H>G$ so the harmonic series is also divergent.

Now let's start again with the harmonic series.

$$H=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$

First we re-arrange the sum, separating the terms whose denominator's are divisible by 2, but leaving 1/2.

$$H=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots$$
$$\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots$$

Note that each term in the second part of the sum is less than it's corresponding term in the first.

$$H=\left|\begin{array}{llllll}\frac{1}{1}+ & \frac{1}{2}+ & \frac{1}{3}+ & \frac{1}{5}+ & \frac{1}{7}+ & \frac{1}{9}+\cdots \\ \frac{1}{4}+ & \frac{1}{6}+ & \frac{1}{8}+ & \frac{1}{10}+\cdots & & \end{array}\right.$$

But we can see that the second sum is $\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\cdots=\frac{1}{2}\left(\frac{1}{1}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)$ which much diverge for the same reason that the harmonic series does.

So the first sum begin greater must also diverge.

$$H_2=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots$$

We repeat this process now taking out the terms whose denominator's are divisible by 3, but leaving 1/3.

Continuing this process, which is equivalent to the Sieve of Eratosthenes, leaving just the sum of the reciprocals of all primes.

So we have shown that this sum is divergent.

Which shows that the number of primes must be infinite, for were it not, the sum would be finite.