ASP-110 - Conic Sections and Transformations

*Take Attendance
*Collect Homework

We're going to talk today about Conic Sections and Transformations.

First, can anyone tell me what a conic section is.

(List the main conic sections, Circle, Parabola, Ellipse, Hyperbola, mention that there are others called degenerate conic sections, eg. a point)

But what is a conic section?

(Among other things it's the curves you get when you intersect a double-cone with a plane.)

Review Grapher Pictures

It's also the case that all conic sections are graphs of second degree equations in x and y.

What's a second degree equation in x and y?

(see what they have to say)

It's an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

We're going to start by looking at an equation that you should already understand.

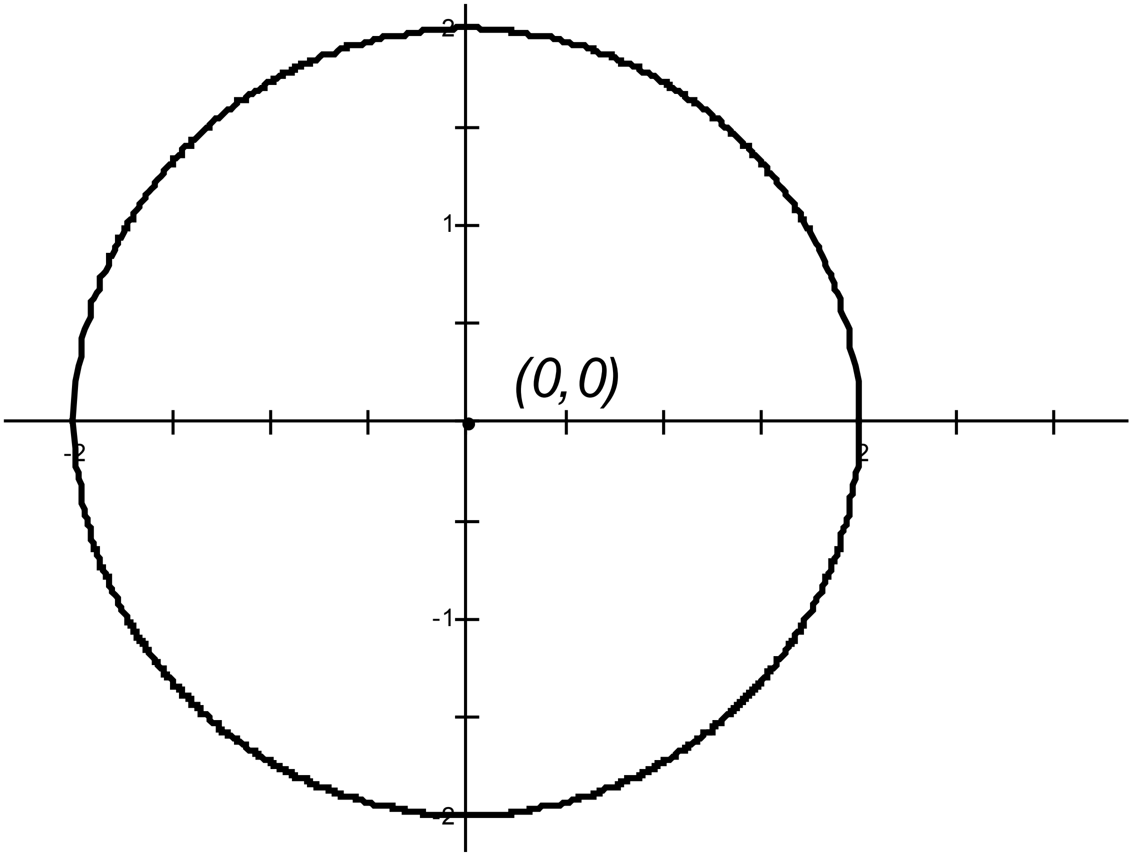
If $B = D = E = 0$ and F=-4 you end up with the following:

$$x^2 + y^2 = 4$$

What is this?

(A circle, radius 4)

Where is the center of this circle? (0,0)

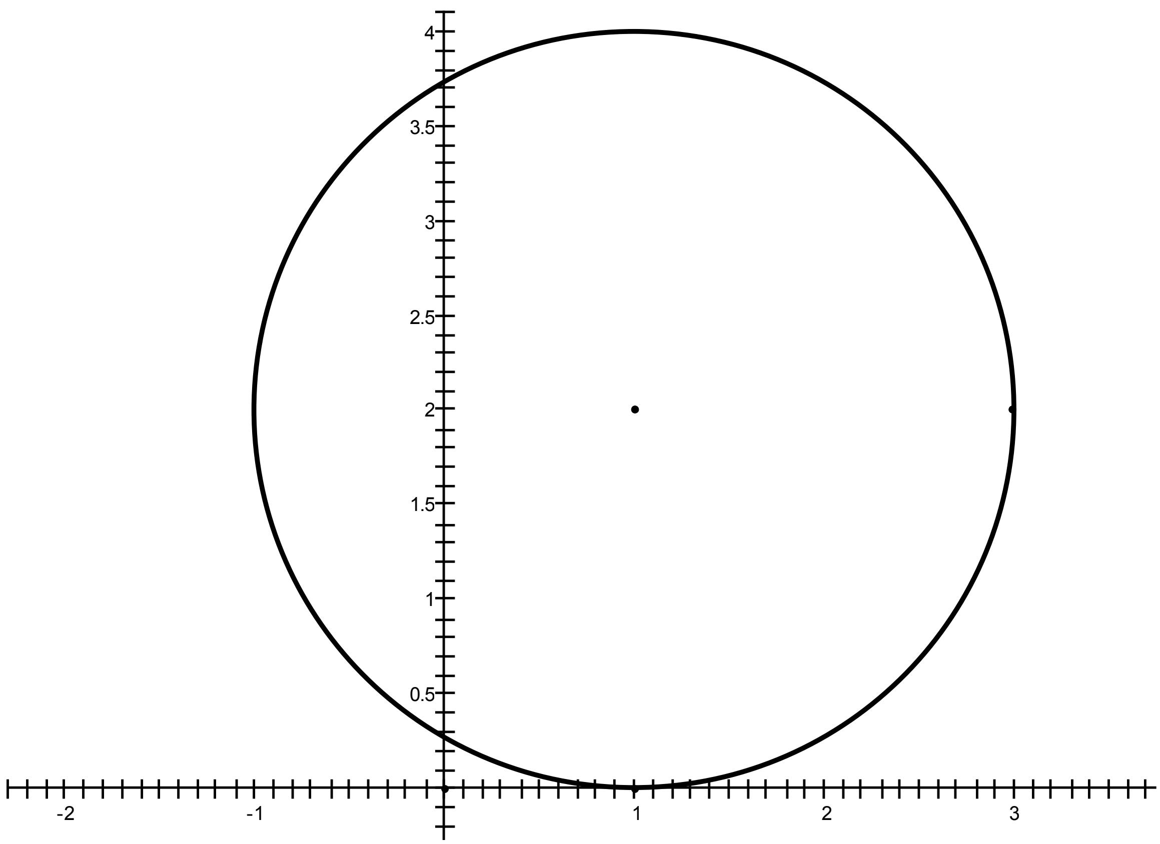

2
1
(0,0)
-2
2
-1
-2

If I change the equation to

$$(x-1)^2 + (y-2)^2 = 4$$

what happens?

(The center moves to (1,2))



This is called a linear transformation.   In particular it is a translation.
We've moved the circle, but we haven't changed it in any other way.

There are other types of linear transformations.

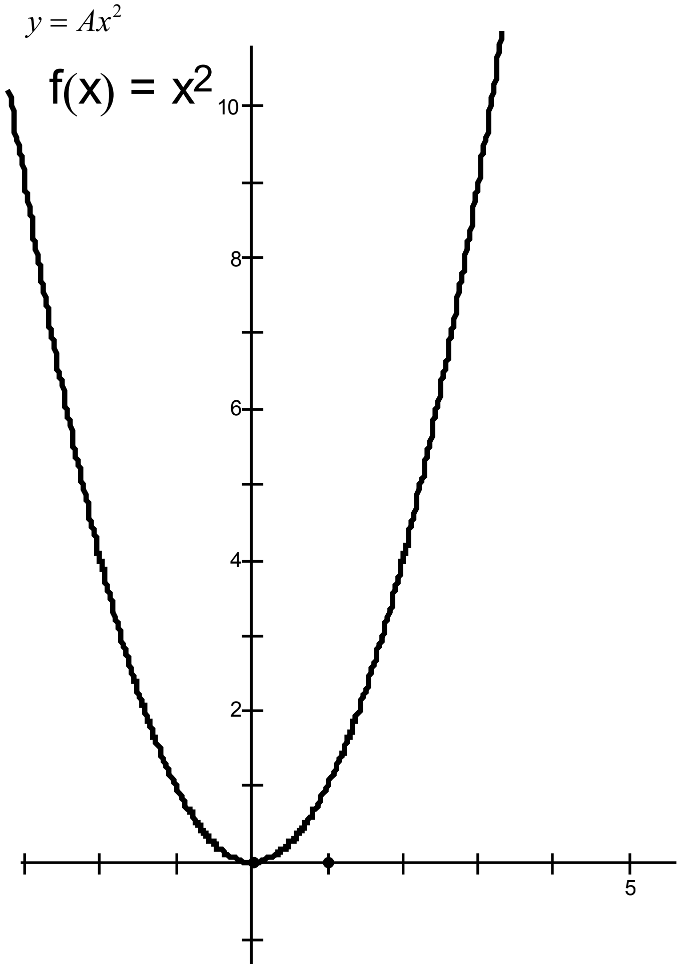
Does anyone know of any?

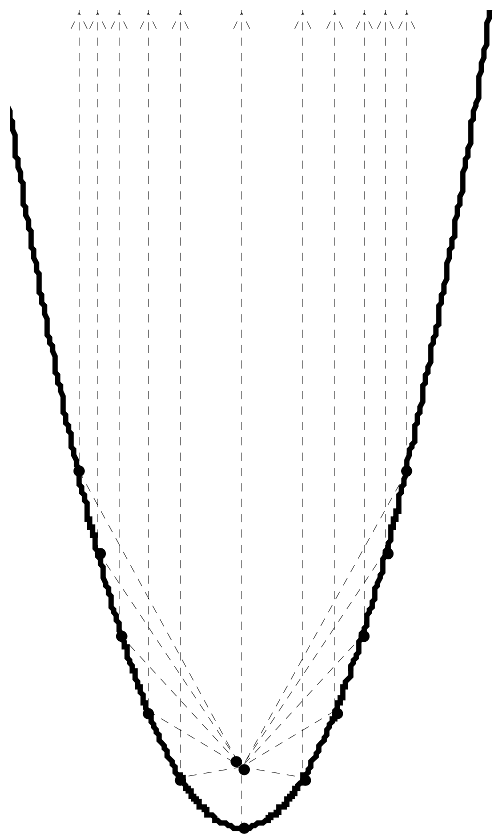
(Dilations, Rotations)

Let's go over the remaining types of Conic Sections:

There's a parabola which could have the equation

$y = Ax^2$

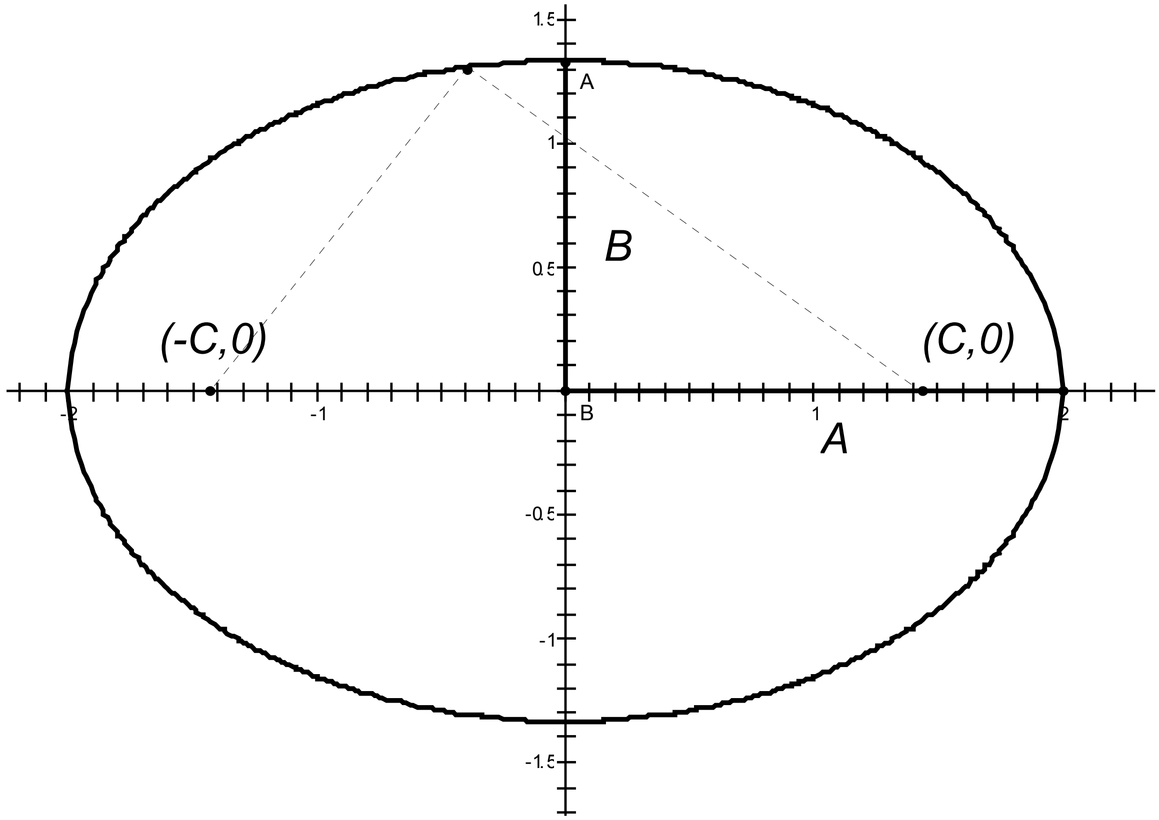




An important property for telescope mirrors is that parallel lines entering the parabola all reflect to the same point called the focus.

There's an Ellipse which could have the equation

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$$



Ellipses have some very cool properties.

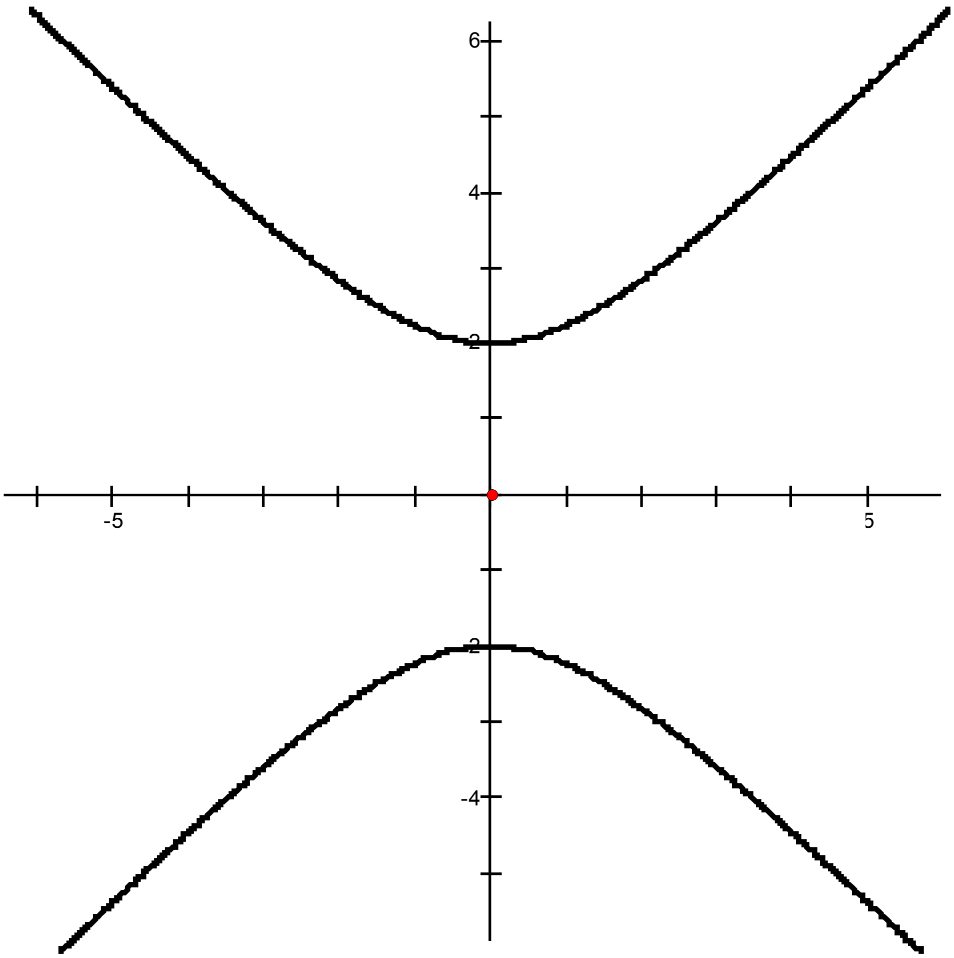
The orbits of planets around the sun are ellipses, not circles.

Ellipses have two points called foci. The sum of the distances from any point on the ellipse to the foci is a constant.
If $A>B$ then this distance is $2A$, otherwise it is $2B$.

If you make a pool table in an elliptical shape, and you hit a ball across one focus, it will bounce and roll across the other.

Finally there's the Hyperbola which could have the equation.

$$\frac{x^2}{A^2} - \frac{x^2}{B^2} = 1$$



Hyperbola's are also used in optics to make telescopes.

They are also the path that a comet that never returns follows.

Determining which Conic Section

If you look at an equation in the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

In order to determine which conic it represents, you need to know how to complete the square.

Does anyone remember how to solve a quadratic equation by completing the square?

We use the pattern $(A+B)^2 = A^2 + 2AB + B^2$
So if we know A and we know $2AB$, we can find

$$B^2 = \left(\frac{2AB}{2}\right)^2$$

Example:

$$x^2 + 6x + 5 = 0$$
$$\left(\frac{6}{2}\right)^2 = 9$$
$$x^2 + 6x + 5 + 4 = 4$$
$$x^2 + 6x + 9 = 4$$
$$(x+3)^2 = 2^2$$
$$x + 3 = \pm\sqrt{2^2} = \pm 2$$
$$x = -3 \pm 2 = -1, -5$$

You can check this with the quadratic formula

$$x = \frac{-6 \pm \sqrt{36-20}}{2} = \frac{-6 \pm \sqrt{16}}{2} = \frac{-6 \pm 4}{2} = -3 \pm 2 = -1, -5$$

So if we have the following equation:

$$4x^2 + 9y^2 + 8x - 18y - 23 = 0$$
$$4x^2 + 8x + \boxed{?} + 9y^2 - 18y + \boxed{?} - 23 = 0$$
$$4\left(x^2 + 2x + \boxed{?}\right) + 9\left(y^2 - 2y + \boxed{?}\right) - 23 = 0$$

What are both the missing numbers?

(1)

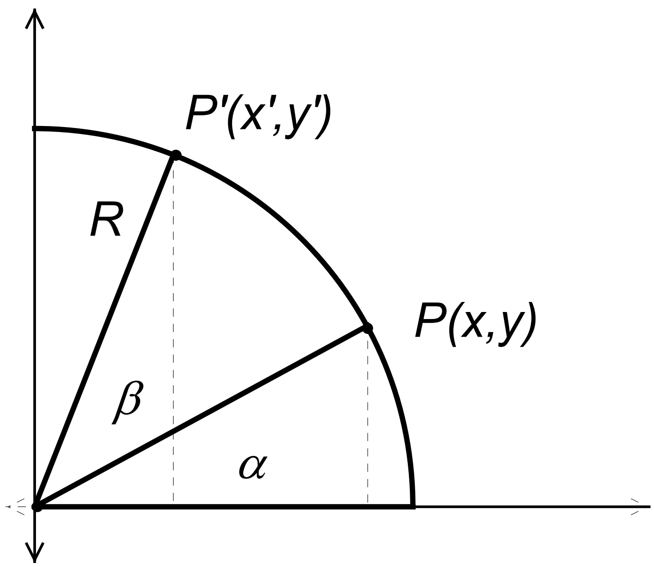
$$4x^2 + 9y^2 + 8x - 18y - 23 = 0$$
$$4x^2 + 8x + \boxed{?} + 9y^2 - 18y + \boxed{?} - 23 = 0$$
$$4\left(x^2 + 2x + 1\right) + 9\left(y^2 - 2y + 1\right) - 23 = 4 + 9$$
$$4\left(x+1\right)^2 + 9\left(y-1\right)^2 = 36$$
$$\frac{\left(x+1\right)^2}{3^2} + \frac{\left(y-1\right)^2}{2^2} = 1$$

So this is an ellipse with center at (-1,1) with a semi-major axis (half the width) of 3 and a semi-minor axis (half the height) of 2.

If there is time....

Rotations

We're going to look at one more type of linear transformation, a rotation.
Consider how we can rotate a point *P* with coordinates (x, y) a distance of R from the origin, around the origin to a new point *P*' with coordinates (x', y').



We know from trigonometry that

$$x = R\cos\alpha$$
$$y = R\sin\alpha$$

We also know that

$$x' = R\cos(\alpha + \beta)$$
$$y' = R\sin(\alpha + \beta)$$

Who remembers how to expand these last two equations using the addition formula?

So we have

$$x' = R\cos(\alpha+\beta) = R\cos\alpha\cos\beta - R\sin\alpha\sin\beta$$
$$y' = R\sin(\alpha+\beta) = R\sin\alpha\cos\beta + R\cos\alpha\sin\beta$$

Notice that

$$x' = [R\cos\alpha]\cos\beta - [R\sin\alpha]\sin\beta$$
$$y' = [R\sin\alpha]\cos\beta + [R\cos\alpha]\sin\beta$$

which we can plug in from our previous equations.

$$x' = x\cos\beta - y\sin\beta$$
$$y' = y\cos\beta + x\sin\beta$$

We usually write this

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

If we treat (x, y) as a column vector $\begin{pmatrix} x \\ y \end{pmatrix}$

We can write this as matrix multiplication

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

A word on rotations as they apply to video games.

A 3D video game does a lot of rotating of points or pixels.

It has to do millions of these a second in order to provide you with a good sense of immersion.

The computer programmers who are writing the code create a 3D rotation matrix that is handed over to special hardware that does all the calculations. That's why you are always reading about the fancy video card a new computer comes with.

The last thing we want to see is what happens when we rotation a conic section using these transformation equations.

Let's start with the ellipse

$$\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$$

And rotate it 45° or $\frac{\pi}{4}$ in radian measure.

Since $\sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$$x'=\frac{1}{\sqrt{2}}(x-y)$$
$$y'=\frac{1}{\sqrt{2}}(x+y)$$

Plugging in we get

$$\frac{x^2-2xy+y^2}{18}+\frac{x^2+2xy+y^2}{8}=1$$

Multiplying by 36 we get

$$4x^2-8xy+4y^2+9x^2+18xy+9y^2=72$$
$$13x^2+10xy+13y^2-72=0$$

Does anyone notice anything usual about this equation?

(It has an *xy* term!)

2nd degree equations with an *xy* term are conic sections rotated!

If you are interested you can un-rotate the equation using the rotation transformation equations with the following equation for the angle.

$$\tan 2\theta=\frac{B}{A-C}$$