

**Integer Exponents**

We define an integer exponent  $n$  over a base  $a \neq 0$  as follows:

$$a^n = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n \text{ times}}$$

So if we multiply two exponential expressions with the same base

$$a^n a^m = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n \text{ times}} \times \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{m \text{ times}} = \underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n+m \text{ times}} = a^{n+m}$$

$$(1) \quad \underline{a^n a^m = a^{n+m}}$$

Similarly, if  $n > m$  we can divide

$$\frac{a^n}{a^m} = \frac{\underbrace{a \cdot a \cdot a \cdots a \cdot a}_{n \text{ times}}}{\underbrace{a \cdot a \cdot a \cdots a \cdot a}_{m \text{ times}}} = \left\{ \frac{a}{a} \cdot \frac{a}{a} \cdot \frac{a}{a} \cdots \frac{a}{a} \right\}_{(m \text{ times})} \cdot \underbrace{a \cdot a \cdot a \cdots a}_{(n-m) \text{ times}} = a^{n-m}$$

$$(2) \quad \underline{\frac{a^n}{a^m} = a^{n-m}}$$

Note that  $\frac{a^n}{a^n} = 1$  so applying the previous equation we see that  $1 = \frac{a^n}{a^n} = a^{n-n} = a^0$

$$(3) \quad \underline{\text{for } a \neq 0 \quad a^0 = 1}$$

Using this preceding

$$\frac{1}{a^n} = \frac{a^0}{a^n} = a^{0-n} = a^{-n}$$

$$(4) \quad \underline{\frac{1}{a^n} = a^{-n}}$$

### Other properties of exponents

$$(a^n)^m = \underbrace{\left( \begin{array}{c} a \cdot a \cdot a \cdots a \cdot a \\ n\text{-times} \end{array} \right)}_{m\text{-times}} \cdot \left( \begin{array}{c} a \cdot a \cdot a \cdots a \cdot a \\ n\text{-times} \end{array} \right) \cdots \left( \begin{array}{c} a \cdot a \cdot a \cdots a \cdot a \\ n\text{-times} \end{array} \right) = \begin{array}{c} a \cdot a \cdot a \cdots a \cdot a \\ nm\text{-times} \end{array} = a^{nm}$$

$$\underline{(5) (a^n)^m = a^{nm}}$$

$$(ab)^n = \begin{array}{c} ab \cdot ab \cdots ab \\ n\text{-times} \end{array} = \begin{array}{c} a \cdot a \cdots a \\ n\text{-times} \end{array} \times \begin{array}{c} b \cdot b \cdots b \\ n\text{-times} \end{array} = a^n b^n$$

$$\underline{(6) (ab)^n = a^n b^n}$$

$$\left( \frac{a}{b} \right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdots \frac{a}{b} = \frac{a \cdot a \cdot a \cdots a}{b \cdot b \cdot b \cdots b} = \frac{a^n}{b^n}$$

$$\underline{(7) \left( \frac{a}{b} \right)^n = \frac{a^n}{b^n}}$$

$$\left( \frac{a}{b} \right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{\frac{1}{a^n}}{\frac{1}{b^n}} = \frac{b^n}{a^n} = \left( \frac{b}{a} \right)^n$$

$$\underline{(8) \left( \frac{a}{b} \right)^{-n} = \left( \frac{b}{a} \right)^n}$$

$$\frac{a^{-n}}{b^{-m}} = \frac{\frac{1}{a^n}}{\frac{1}{b^m}} = \frac{b^m}{a^n}$$

$$\underline{(9) \frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}}$$

## Summary

$a^n a^m = a^{n+m}$	$\frac{a^n}{a^m} = a^{n-m}$	for $a \neq 0$ $a^0 = 1$
$\frac{1}{a^n} = a^{-n}$	$(a^n)^m = a^{nm}$	$(ab)^n = a^n b^n$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	$\frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n}$

## Example

Simplify  $\left(\frac{x}{y}\right)^3 \left(\frac{y^3 x}{z}\right)^4$

$$\left(\frac{x}{y}\right)^3 \left(\frac{y^3 x}{z}\right)^4 = \frac{x^3}{y^3} \cdot \frac{(y^3)^4 x^4}{z^4} = \frac{x^3 y^{12} x^4}{y^3 z^4} = \frac{x^{3+4} y^{12-3}}{z^4} = \frac{x^7 y^9}{z^4}$$

## Fractional exponents

We start with the question, what does  $5^{1/2}$  mean?

If  $x = 5^{1/2}$  then  $x \cdot x = 5^{1/2} \cdot 5^{1/2} = 5^{1/2+1/2} = 5^1 = 5$

So, we have  $x^2 = 5$  so  $x = \sqrt{5}$

Similarly

$$(1) \quad \underline{a^{1/n} = \sqrt[n]{a}}$$

What then is  $a^{n/m}$ ?

$$a^{n/m} = (a^n)^{1/m} = \sqrt[m]{a^n}$$

but also

$$a^{n/m} = (a^{1/m})^n = (\sqrt[m]{a})^n$$

$$(2) \quad \underline{a^{n/m} = \sqrt[m]{a^n} = (\sqrt[m]{a})^n}$$

## Other properties of $n^{\text{th}}$ roots

$$\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n} b^{1/n} = \sqrt[n]{a} \sqrt[n]{b}$$

$$(3) \quad \underline{\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}}$$

$$\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$(4) \quad \underline{\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}}$$

$$\sqrt[n]{\sqrt[m]{a}} = \sqrt[n]{a^{1/m}} = (a^{1/m})^{1/n} = a^{1/mn} = \sqrt[nm]{a}$$

$$(5) \quad \underline{\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}}$$

Lastly, consider  $\sqrt[n]{a^n}$  where  $n$  is even and  $a \in \mathbb{Z}$ .

Since  $n$  is even,  $a^n \geq 0$ , so  $\sqrt[n]{a^n} \geq 0$  but  $\sqrt[n]{a^n} = a^{n/n} = a^1 = a$  so

(6)  $\sqrt[n]{a^n} = |a|$  for  $n$  even

### Summary

$a^{1/n} = \sqrt[n]{a}$	$a^{n/m} = \sqrt[m]{a^n} = (\sqrt[m]{a})^n$	$\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$
$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$	$\sqrt[n]{a^n} =  a $ for $n$ even