Section 1-4, Mathematics 108

Rational Expressions

A rational expression is a fraction in which the numerator and the denominator are polynomials.

Examples:

$$\frac{2x}{x-1} \qquad \frac{y-2}{y^2+4} \qquad \frac{x^3-x}{x^2-5x+6}$$

Whereas

$$\frac{x}{\sqrt{x+1}}$$

is not because it has a radical

Domain of a Rational Expression

What we mean by the domain is those real values that we can plug into the expression. We can define a restricted domain

Example:

$$\frac{2x}{x-1} \quad \text{D} = [5,6]$$

However in the absence of such a definition by default we mean all real numbers for which the expression is defined.

Example:

$$\frac{2x}{x-1} \quad \text{D} = \{x : x \in \mathbb{R} \; and \; x \neq 1\}$$

Example:

$$\frac{x}{x^2-x-30} = \frac{x}{(x-6)(x+5)} \quad \text{so D} = \{x : x \in \mathbb{R} \; and \; x \neq -5 \; and \; x \neq 6\}$$

Simplifying Rational Expressions

If $C \neq 0$ then $\frac{AC}{BC} = \frac{A}{B}$

Example:

So we have

$$\frac{x^2 - 1}{x^2 + x - 2} = \frac{(x+1)(x-1)}{(x+2)(x-1)} = \frac{(x+1)}{(x+2)} \text{ where } x \neq 1$$

Note that after simplification we cannot increase the domain

Multiplying Rational Expressions

With polynomials like numbers $\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}$

Example:

$$\frac{(x^2 + 2x - 3)}{(x^2 + 8x + 16)} \cdot \frac{3x + 12}{x - 1} = \frac{(x+3)(x-1)}{(x+4)^2} \cdot \frac{3(x+4)}{x-1} = 3\frac{x+3}{x+4}$$

But with $x \neq 1, -4$

Dividing Rational Expressions

$$\frac{A}{B} \div \frac{C}{D} = \frac{AD}{BC}$$

Example:

$$\frac{x-4}{x^2-4} \div \frac{x^2 - 3x - 4}{x^2 + 5x + 6} = \frac{x-4}{x^2-4} \cdot \frac{x^2 + 5x + 6}{x^2 - 3x - 4} =$$

$$\frac{x-4}{(x-2)(x+2)} \cdot \frac{(x+2)(x+3)}{(x-4)(x+1)} = \frac{x+3}{(x-2)(x+1)} = \frac{x+3}{x^2 - x + 2}$$

But with $x \neq 2, -2, 4, -1$

Adding Rational Expressions
But with $x \neq 1, -4$

Just like with numbers, you can only add rational expressions if they have the same denominator.

$$\frac{A}{C}+\frac{B}{C}=\frac{A+B}{C}$$

But $\frac{A}{C}+\frac{B}{D}=\frac{AD}{CD}+\frac{BC}{CD}=\frac{AD+BC}{CD}$

Example:

$$\frac{3}{x-1}+\frac{x}{x+2}=\frac{3(x+2)}{(x-1)(x+2)}+\frac{x(x-1)}{(x+2)(x-1)}=$$
$$\frac{3(x+2)+x(x-1)}{(x-1)(x+2)}=\frac{3x+6+x^2-x}{(x-1)(x+2)}=\frac{x^2+2x+6}{(x-1)(x+2)}$$

Compound Expressions

$$\frac{\frac{x}{y}+1}{1-\frac{y}{x}}=\frac{\frac{x+y}{y}}{\frac{x-y}{x}}=\frac{x+y}{y}\cdot\frac{x}{x-y}$$

Rationalizing a Denominator

We recall that

$$(A+B)(A-B)=A^2-B^2$$

Example:

$$\frac{1}{1+\sqrt{2}}=\frac{1}{1+\sqrt{2}}\cdot\frac{1-\sqrt{2}}{1-\sqrt{2}}=\frac{1-\sqrt{2}}{1^2-\left(\sqrt{2}\right)^2}=\frac{1-\sqrt{2}}{1-2}=\frac{1-\sqrt{2}}{-1}=\sqrt{2}-1$$