Section 3-5, Mathematics 108

Complex Zeros and the Fundamental Theorem of Algebra

Fundamental Theorem of Algebra

This theorem seems strangely incomplete.

"Every polynomial has a complex root"

Note that by a complex root we mean either a real number, an imaginary number or a complex number. Real and imaginary numbers are also complex numbers.

To complete theorem, consider a polynomial of degree n with root c.

Since c is a root of the polynomial,

$$P(c) = 0$$

The factor theorem tells us that $(x-c)$ is a factor of $P(x)$
meaning that there is a polynomial $Q(x)$ such that

$$P(x) = (x-c)Q(x)$$

We know that $Q(x)$ will have degree n-1.

So we can repeat this process n times finding n roots of $P(x)$

This gives us our next theorem

The Complete Factorization Theorem

A polynomial of degree n will have n complex roots.

Factoring a Polynomial by Grouping

Example

$$P(x) = x^3 - 3x^2 + x - 3$$

We group the first two and the 2nd two terms

$$P(x) = \left(x^3 - 3x^2\right) + \left(x - 3\right)$$

and factor out x^2 from the first group

$$P(x) = x^2\left(x - 3\right) + \left(x - 3\right)$$

Here we see we can factor out (*x*-3)

$$P(x) = \left(x - 3\right)\left(x^2 + 1\right)$$

The factor $\left(x^2 + 1\right)$ is irreducible in just real numbers, however it is easily factored using complex numbers

$$\left(x^2 + 1\right) = \left(x + i\right)\left(x - i\right)$$

So finally we have

$$P(x) = \left(x - 3\right)\left(x + i\right)\left(x - i\right)$$

And so the three roots are 3, *i*, and -*i*

Factoring a Polynomial using the Rational Root Theorem

Example

$P(x) = x^3 - 2x + 4$

The rational root theorem tells us that the possible roots are

$\pm 1, , \pm 2, \pm 4$

By trial and error we find that -2 is a root.

We divide using synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

So $x^3 - 2x + 4 = (x+2)(x^2 - 2x + 2)$

At this point we use the quadratic formula to find the roots of $x^2 - 2x + 2$

$$\frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

So the complete factorization is $P(x) = (x+1)(x-1+i)(x-1-i)$

Multiplicity of Zeros

Although a polynomial of degree *n* will have *n* roots, they do not have to be distinct.

As a trivial example

$(x-1)^5$ has 5 roots, but they are all 1.

We say the polynomial has a root 1 of **multiplicity** 5.

Example

$P(x)=3x^5+24x^3+48x$

First we factor out 3*x*

$$3x^5+24x^3+48x=3x\left(x^4+8x^2+16\right)$$

The left factor is a disguised quadratic in the form of a perfect square, so we get

$$3x\left(x^4+8x^2+16\right)=3x\left(x^2+4\right)^2$$

Further factoring gives up

$$3x\left(x^4+8x^2+16\right)=3x(x+2i)(x-2i)(x+2i)(x-2i)=3x(x+2i)^2(x-2i)^2$$

So the polynomial has 5 roots including 0, 2*i,* and -2*i* but 2*i* and -2*i* each have multiplicity 2.

Finding a polynomial with specified roots

Finding a polynomial with specified roots is straight forward.

If we have a list of roots $c_1, c_2, \ldots c_n$ then our polynomial is just the product of a constant and *x* minus each root

$$P(x) = a(x - c_1)(x - c_2)\ldots(x - c_n)$$

Example

Find a polynomial with roots *i,* -*i*, 2, -2 where $P(3) = 25$

$$P(x) = a(x - i)(x + i)(x - 2)(x + 2) =$$

$$a(x^2 + 1)(x^2 - 4) =$$

$$a(x^4 - 3x^2 - 4)$$

Since we know $P(3) = 25$

$$P(3) = a(81 - 27 - 4) = a50 = 25$$

so

$$a = \frac{1}{2}$$

$$P(x) = \frac{1}{2}(x^4 - 3x^2 - 4) = \frac{1}{2}x^4 - \frac{3}{2}x^2 - 2$$

Conjugate zeros theorem

You may have noticed that complex roots seem to come in pairs

$a+bi$ and $a-bi$

These are call complex conjugates, or just conjugates when the context is clear.

The conjugate zeros theorem states that when a polynomial has real coefficients, that any complex roots will come as conjugate pairs.

Example

Find a polynomial with real coefficients that has roots 1/2 and 3-*i*.

Since the polynomial has real coefficients and a root 3-*i* it must also have it's complex conjugate 3+*i*.

$$P(x)=(2x-1)\left(x-(3-i)\right)\left(x-(3+i)\right)$$

Using Foil

$$\left(x-(3-i)\right)\left(x-(3+i)\right)=x^2-(3+i)x-(3-i)x+\left(3^2+1\right)=$$

$$x^2-6x+10$$

Then

$$(2x-1)\left(x^2-6x+10\right)=2x^3-12x^2+20x+{}^{-}x^2+6x-10=$$

$$2x^3-13x^2+26x-10$$

Linear & Quadratic Factors Theorem.

A quadratic polynomial that has no real zeros is called **irreducible**.

The linear and quadratic factors theorem says that every polynomial with real coefficients can be factored into a product of linear (*x*-c) factors and irreducible quadratic factors.

Example

$P(x) = x^4 - 2x^2 - 8$

This is a disguised quadratic that we can reverse foil

$x^4 - 2x^2 - 8 = (x^2 + 2)(x^2 - 4) = (x^2 + 2)(x + 2)(x - 2)$

Note that we have factored to two linear factors
$(x+2)(x-2)$ and an irreducible factor $(x^2 + 2)$

The irreducible factor can be further factored only with complex numbers into

$(x^2 + 2) = (x + \sqrt{2}i)(x - \sqrt{2}i)$