Trigonometry 11 Mathematics 108

Identities

What is an identity?

An identity is an equation showing an equivalence between two expressions for all values of a variable.

Example:

x + x - 6 = (4x - 12)/ 2

To show the first two are equivalent we state a theorem:

A = 2(x-3) if and only if A = x + x - 6

We start with 2(x-3) and manipulate it until we end up with x + x - 6 as follows:

2(x-3) = 2x - 2(3) = x + x - 6, To prove the only if part we would have to start with

x + x - 6 and show 2(x-3) but in this example, we can just state that the steps are reversible.

Let's review some basic trigonometric equivalences.

$$\tan x = \frac{\sin x}{\cos x} \quad \cot x = \frac{\cos x}{\sin x}$$

$$\csc x = \frac{1}{\sin x} \quad \sec x = \frac{1}{\cos x}$$

A simple Identity Proof

Using these, let's try a simple proof of an identity:

$$\tan x = \frac{1}{\cot x}$$

The steps would look like this:

$$\tan x = \frac{\sin x}{\cos x} \text{ (definition}$$
$$\frac{\sin x}{\cos x} = \frac{\sin x}{\cos x} \times \frac{1/\sin x}{1/\sin x} \text{ (multiplication by 1}$$

$$\frac{\sin x}{\cos x} \times \frac{1/\sin x}{1/\sin x} = \frac{1}{\cos x/\sin x} \text{(rules for fractions}$$

$$\frac{1}{\cos x/\sin x} = \frac{1}{\cot x} \text{(definition}$$

This shows $\tan x = \frac{1}{\cot x}$

To show $\frac{1}{\cot x} = \tan x$ we can reverse the steps.

Or similarly we can just state that "The steps are reversible".

Pythagorean Identities

Previously we demonstrated the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1$$

This identity leads to two more identities

$$\sin^2 x + \cos^2 x = 1$$
$$\left(\sin^2 x + \cos^2 x\right)\frac{1}{\cos^2 x} = 1 \cdot \frac{1}{\cos^2 x}$$
$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$
$$\tan^2 x + 1 = \sec^2 x$$

$$\sin^2 x + \cos^2 x = 1$$
$$\left(\sin^2 x + \cos^2 x\right)\frac{1}{\sin^2 x} = 1 \cdot \frac{1}{\sin^2 x}$$
$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}$$
$$1 + \cot^2 x = \csc^2 x$$

These three identities are all referred to as Pythagorean identities.

Even-Odd Identities

We've previously see that

$\sin(-x) = -\sin(x)$ showing sine to be an odd function and

$\cos(-x) = \cos(x)$

Showing cosine to be an even function.

Let's check the other 4 trigonometric functions

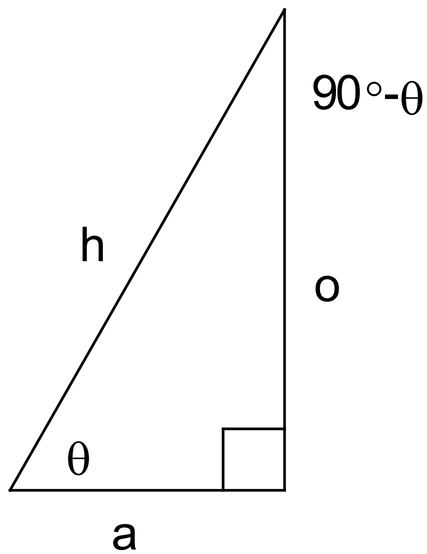
$\tan(-x) = \dfrac{\sin(-x)}{\cos(-x)} = \dfrac{-\sin(x)}{\cos(x)} = -\tan(x)$ so tangent is an odd function

The same steps show co-tangent is also odd

$\sec(-x) = \dfrac{1}{\cos(-x)} = \dfrac{1}{\cos(x)} = \sec(x)$ so secant is an even function

$\csc(-x) = \dfrac{1}{\sin(-x)} = \dfrac{1}{-\sin(x)} = -\csc(x)$ so co-secant is an odd function

Cofunction Identities



This diagram indicates that since

$$\sin\theta = \frac{o}{h}$$

$$\cos\left(90^\circ - \theta\right) = \frac{o}{h}$$

that

$$\sin\theta = \cos\left(90^\circ - \theta\right)$$

and similarly

$$\cos\theta = \sin\left(90^\circ - \theta\right)$$

Using these we can find

$$\tan\left(90^\circ - \theta\right) = \frac{\sin\left(90^\circ - \theta\right)}{\cos\left(90^\circ - \theta\right)} = \frac{\cos\theta}{\sin\theta} = \cot\theta$$

similarly

$$\cot\left(90^\circ - \theta\right) = \tan\theta$$

and

$$\sec\left(90^\circ - \theta\right) = \csc\theta$$

$$\csc\left(90^\circ - \theta\right) = \sec\theta$$

Simplifying Trigonometric Expressions.

When faced with a trigonometric expression, one useful strategy is to rewrite the expression in terms of sines and cosines and then simplify.

$$\cot x + \tan x \sin x = \cos x + \frac{\sin x}{\cos x}\sin x =$$

$$\frac{\cos^2 x + \sin^2 x}{\cos x} = \frac{1}{\cos x} = \sec x$$

Another strategy is combine fractions.

$$\frac{\sin x}{\cos x} + \frac{\cos x}{1+\sin x} = \frac{\sin x\left(1+\sin x\right)+\cos^2 x}{\cos x\left(1+\sin x\right)} =$$

$$\frac{\sin x + \sin^2 x + \cos^2 x}{\cos x\left(1+\sin x\right)} = \frac{\sin x + 1}{\cos x\left(1+\sin x\right)} = \frac{1}{\cos x} = \sec x$$

A note on disproving identities

While proving an expression is an identity requires a proof,
disproving an identity merely requires a single case.

You might suspect that $\sin x + \cos x = 1$ is an identity since it is true for 0° *and* 90°

But for 180° we get $\sin 180^\circ + \cos 180^\circ = 0 + {}^{-}1 = {}^{-}1 \neq 1$

When proving an identity, performing the same operation on each side is not valid unless both operations are reversible.

So for example

$$\sin(-x) = \sin(x)$$
$$-\sin(x) = \sin(x)$$
$$(-\sin x)^2 = (\sin x)^2$$
$$-1^2 \sin^2 x = \sin^2 x$$
$$\sin^2 x = \sin^2 x$$

is not a valid proof, and the original expression is not an identity.

Example

$$\cos\theta\left(\sec\theta-\cos\theta\right)=\sin^2\theta$$

$$\cos\theta\left(\frac{1}{\cos\theta}-\cos\theta\right)=1-\cos^2\theta=$$

$$\sin^2\theta+\cos^2\theta-\cos^2\theta=\sin^2\theta$$

Example

$$2\tan x\sec x=\frac{1}{1-\sin x}-\frac{1}{1+\sin x}$$

$$\frac{1}{1-\sin x}-\frac{1}{1+\sin x}=\frac{1+\sin x-\left(1-\sin x\right)}{\left(1-\sin x\right)\left(1+\sin x\right)}=$$

$$\frac{2\sin x}{1-\sin^2 x}=\frac{2\sin x}{\sin^2 x+\cos^2 x-\sin^2 x}=$$

$$\frac{2\sin x}{\cos^2 x}=2\frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}=2\tan x\sec x$$

Example

$$\frac{\cos x}{1-\sin x}=\sec x+\tan x$$

$$\frac{\cos x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}=\frac{\cos x+\cos x\sin x}{1-\sin^2 x}=$$

$$\frac{\cos x+\cos x\sin x}{\cos^2 x}=\frac{1}{\cos x}+\frac{\sin x}{\cos x}=\sec x+\tan x$$

You can however work both sides of the identity.
Once you find the two sides equal, you can use the fact that your steps are reversible to finish the proof.

$$\frac{1+\cos x}{\cos x} = \frac{\tan^2 x}{\sec x - 1}$$

$$\frac{1}{\cos x} + 1 = \frac{\sec^2 x - 1}{\sec x - 1}$$

$$\sec x + 1 = \frac{(\sec x - 1)(\sec x + 1)}{\sec x - 1}$$

$$\sec x + 1 = \sec x + 1$$

Note that since these steps are reversible, the proof is valid

7.1 13-18, 37, 39, 42, 50, 64