Math 109 Calc 1 Lecture 16

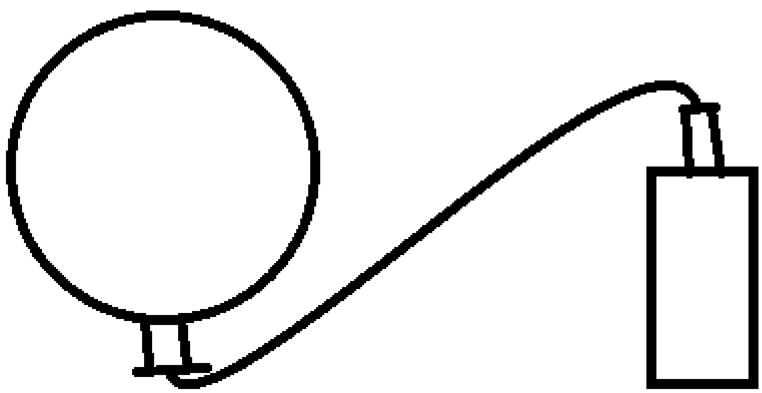
Related Rates

Section 3.9

Related Rates

To demonstrate how related rates problems work, it is best to view a few examples. The process is to use the chain rule and apply it to the geometry of the problem.

Example



We have an example of a related rates problems here. We have two rates that are related, but in a non-simple way.

We have a balloon being filled with gas at a constant rate of $100cm^3/sec$. We want to know how fast the radius is increasing when the radius is 25cm.

The first rate that we know is the rate at which the volume of the balloon is increasing. We know that the rate that the radius is increasing is related to the rate of the volume.

In most such cases it will be useful to look at the chain rule in the Leibniz form.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

In our example it will be better to state it this way.

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

We know from the description of the problem that $\frac{dV}{dt} = 100cm^3/sec$.
To find $\frac{dr}{dt}$ we just need to know $\frac{dV}{dr}$, the rate of change of Volume with respect to radius.

To find $\frac{dV}{dr}$ we need the relationship between V and r?

Here we need to rely on our former knowledge of geometry:

$$V = \frac{4}{3}\pi r^3$$

Using the power rule we see that

$$\frac{dV}{dr} = 4\pi r^2$$

So, our original equation becomes

$$100cm^3/sec = 4\pi r^2 \frac{dr}{dt}$$

Or

$$\frac{dr}{dt} = \frac{100cm^3/sec}{4\pi r^2}$$

Let's do a sanity test on this answer to see if it makes sense.

With a constant volume increase, as the radius increases, the rate that it increases decreases. That is what we would expect.

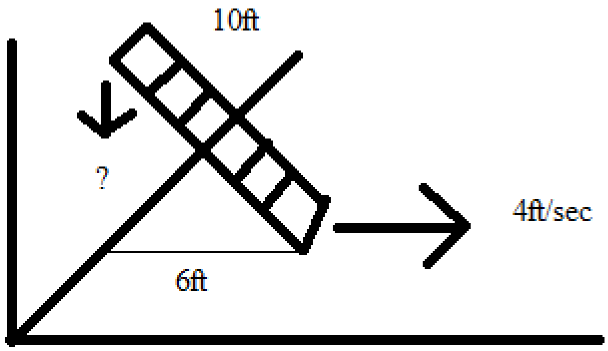
The problem requests $\frac{dr}{dt}$ when r=50cm so we plug in

$$\frac{dr}{dt} = \frac{100cm^3/sec}{4\pi(25cm)^2} = \frac{1}{25\pi}cm/sec$$

Note the use of units confirming a reasonable answer. We expect dr/dt to be in units of distance per time.

Example

A 10ft long ladder rests against a vertical wall. The bottom slides out at a rate of 4ft/sec. At what rate is the top of the ladder falling when the bottom is 6ft from the wall?



Using the usual coordinate system, let the height of the ladder be *y* and the distance to the bottom of the ladder from the wall *x*.

Here we want to find requests $\frac{dx}{dt}$.

The relationship between *x* and *y* can be described using the Pythagorean theorem:

$$x^2 + y^2 = 10^2$$

Here we can differentiate implicitly with respect to *t*.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Solving for $\frac{dy}{dt}$ we get

$$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

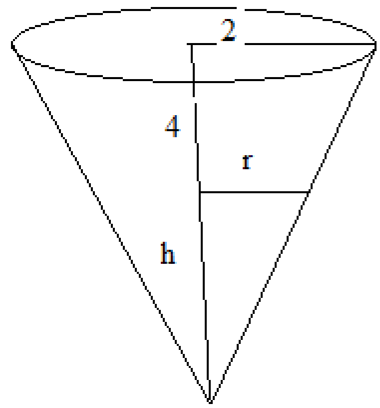
When x = 6ft, by the Pythagorean theorem we have

$6^2 + y^2 = 10^2$ or $y = 8$

So, $\frac{dy}{dt} = -\frac{6}{8} \cdot 4ft/sec \; = \; -3ft/sec$

Note the negative sign indicating that the direction of the top of the ladder is down.

Example



A water tank in the shape of an inverted cone has a base radius of 2 and a height of 4.

Water enters the tank at a rate of $2m^3/sec$.

How fast is the water rising when the water is 3m deep?

Looking at the equation:

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

We see that we have the value of $\frac{dV}{dt}$ and we want the value of $\frac{dh}{dt}$

So, we are left with the task of finding $\frac{dV}{dh}$.

The volume of a cone with height *h* and base radius *r* is

$$V = \frac{1}{3}\pi h r^2$$

To find *V* in terms of just *h* we use the obvious similar triangles in the diagram showing that

$$\frac{4}{2} = \frac{h}{r}$$

or

$$r = \frac{h}{2}$$

Plugging this in we find

$$V = \frac{1}{3}\pi h \left(\frac{h}{2}\right)^2 = \frac{\pi h^3}{12}$$

Differentiating we get

$$\frac{dV}{dh} = \frac{\pi h^2}{4}$$

so

$$\frac{dV}{dt} = 2m^3/sec = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

or

$$\frac{dh}{dt} = \frac{4}{\pi h^2} 2m^3/sec$$

At a depth of 3m we get

$$\frac{dh}{dt} = \frac{4}{\pi (3m)^2} 2m^3/sec = \frac{8}{9\pi}\ m/sec$$

Again, note that the final units are consistent.

In class Examples

A water trough with a vertical cross section in the shape of a n equilateral triangle, one vertex down, is being filled at a rate of $4ft^2$ /min. Given that the trough is 12 feet long, how fast is the water level rising when the water reaches a depth of 1.5 feet?

1.* A point moves along the straight line $x + 2y = 2$. Find: (a) the rate of change of the y-coordinate, given that the x-coordinate increases 4 units/sec; (b) the rate of change of the x-coordinate, given that the y-coordinate decreases 2 units/sec.

2. The volume of a contracting cube is decreasing at the rate of 2 in³/min. Find (a) the rate of change of an edge, and (b) the rate of change of the total surface area when the volume of the cube is 64 in³.

3.* A point P is moving in the circular orbit $x^2 + y^2 = 25$. As it passes through the point (3, 4), its y-coordinate is decreasing at the rate of 2 units/sec. How is the x-coordinate changing?

4. A ladder 13 ft long is leaning against a wall. If the base of the ladder is being pulled away from the wall at the rate of $\frac{1}{2}$ ft/sec, how fast is the top of the ladder being lowered when the base is 5 ft from the wall?

5.* Taking

$$x = -t\sqrt{t+1}, \qquad t \geq 0$$

as the equation of motion, find
(a) the velocity when $t = 1$,
(b) the acceleration when $t = 1$,
(c) the speed when $t = 1$,
(d) the rate of change of the speed when $t = 1$,
(e) the time t, if it exists, when the velocity is a maximum,
(f) the time t, if it exists, when the speed is a maximum.

6.* The height of a cylinder is being increased at the rate of 4 in./min. If the volume of the cylinder is to be kept constant, at what rate must the radius be diminished at each instant?

7. In the special theory of relativity the mass of a particle moving at velocity v is

$$m\left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

where m is the mass at rest and c is the speed of light. At what rate is the mass changing when the particle's velocity is $\frac{1}{2}c$ and the rate of change of the velocity is $0.01c$/sec?

8.* A conical paper cup of radius 2 in. and height 6 in. is leaking water at the rate of 1 in^3/min. At what rate is the level of the water being lowered: (a) when the water is 3 in. deep, (b) when the cup is half full?

9. The shadow cast by a man standing 3 ft from a lamp post is 4 ft long. If the man is 6 ft tall and walks away from the lamp post at a speed of 400 ft/min, at what rate will his shadow be lengthening: (a) a quarter of a minute later, (b) when he is 20 ft from the lamp post?

10* A stone is thrown upward from ground level with an initial speed of 32 ft/sec. (a) How many seconds later will it hit the ground? (b) What will be the maximum height attained? (c) With what initial speed should it be thrown if it is to reach a maximum height of 36 ft?

11. To estimate the height of a bridge a man drops a stone into the water below. How high is the bridge (a) if the stone hits the water 3 seconds later? (b) if the man hears the splash 3 seconds later? (Use 1080 ft/sec as the speed of sound.)

12.* A falling stone is observed to be at a height of 100 ft. Two seconds later it is observed to be at a height of 16 ft. (a) From what height was it dropped? (b) If it was thrown down with an initial speed of 5 ft/sec, from what height was it thrown? (c) If it was thrown upward with an initial speed of 10 ft/sec, from what height was it thrown?