Math 109 Calc 1 Lecture 20

The Mean Value Theorem

Section 4.2

Opinion:

Today we are going to learn about Rolle's Theorem and the Mean Value Theorem. There is an irony involved with these theorems. They are both critical to the subject of mathematical analysis, the area of math that examines the underpinnings of calculus.

The irony is that neither theorem will be very useful to you. I will show you some examples of how they can be used. Mainly however they are important in proving other theorems, something that is not a high priority in this class.

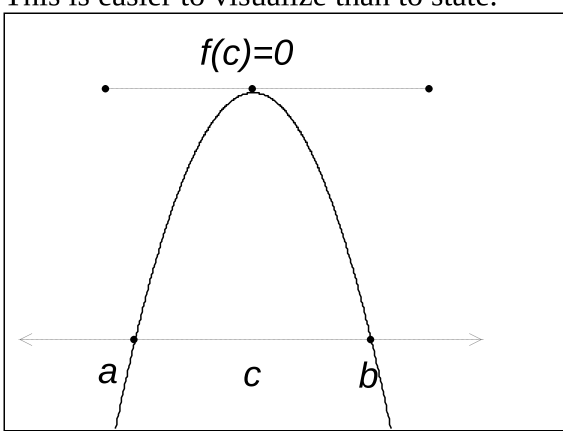
Rolle's Thereon

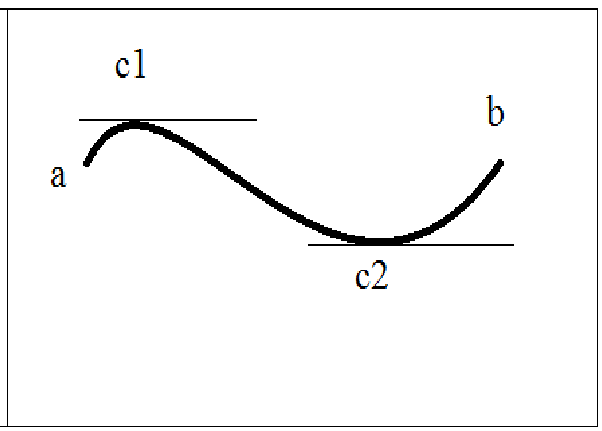
Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval [a,b]
2. f is differentiable on the open interval (a,b)
3. $f(a) = f(b)$

Then there exists at least one number c in (a,b) such that $f'(c) = 0$

This is easier to visualize than to state.





The proof has the following logic.
If we look at the derivative of a point *x* on *f* between *a* and *b* one of these conditions exist.

If it is zero, we are done.

If it is positive, that means the function is increasing at some point.
Since $f(a) = f(b)$ the derivative must be negative at some point.
Since the derivative is continuous, the intermediate value says there must be some *c* where it is zero.

The same logic holds in reverse for a point *x* where the derivative is negative.

Example:

If you throw a ball up in the air, at some moment its velocity is zero.

When you throw the ball up it eventually comes down so $f(t1) = f(t2)$

By Rolle's theorem there is a time *t* where $f'(t) = 0$.

Example:

Prove that $f(x) = x^3 + x - 1$ has **exactly** 1 real root.

Part 1, $f(x)$ has one real root
Since *f* is continuous and $f(1) = 1$ *and* $f(-1) = -3$ by the intermediate value theorem there has to be some *c* on the interval (-1,1) for which $f(c) = 0$.

Part 2: Proof by Contradiction

Now assume that *f* has at least two real roots *a* and *b*.

Then $f(a) = f(b) = 0$.

So, there must be a *c* such that n $f'(c) = 0$.

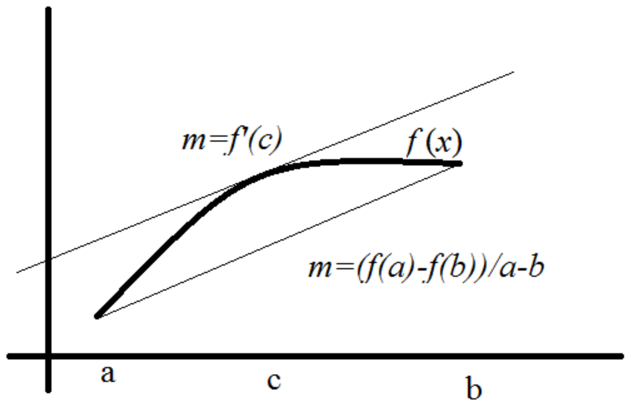
But $f'(x) = 3x^2 + 1$ and $3x^2 + 1 = 0$ has no real roots.

So, we conclude the function has exactly one real root.

The Mean Value Theorem

Let *f* be a function that satisfies the following three hypotheses:

1. *f* is continuous on the closed interval [a,b]
2. *f* is differentiable on the open interval (a,b)
3. Then there is a number c in (a,b) such that $f'(c) \ = \frac{f(a)-f(b)}{a-b}$



In this illustration one can see that the theorem is saying that there is a tangent to the curve between a and b which is parallel to the secant connecting a and b.

Example:

Given the function $f(x) = x^3 - x$ on the interval (0, 2), find *c* such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Since

$f(0) = 0$ and $f(2) = 8 - 2 = 6$

the theorem tells us that there is a *c* in (0,2) for which

$f'(c) = 3c^2 - 1 = \frac{6-0}{2-0} = 3$

So, $c = \pm\sqrt{\frac{4}{3}} = \pm\frac{2}{\sqrt{3}}$

Of course, we are looking for the positive value $0 < \frac{2}{\sqrt{3}} < 2$

What can we do in Calculus with this theorem?

As I mentioned, the mean value theorem is very useful in analysis. Here is a simple example of the types of things we can prove using it.

Theorem:

If $f'(x) = 0$ on the interval (a,b) then If $f'(x)$ is constant on (a,b).

Proof:

Choose two arbitrary numbers x_1 and x_2 with $a < x_1 < x_2 < b$

By the mean value theorem there is a c with $x_1 < c < x_2$ such that

$$f'(c) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}$$

But since $f'(c) = 0$ we have

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = 0$$

$$f(x_1) - f(x_2) = 0$$

$$f(x_1) = f(x_2)$$

So f is a constant on (a,b)