Math 109 Calc 1 Lecture 31

## 5.3 The Fundamental Theorem of Calculus

We have previously discussed the very important connection between Differential and Integral Calculus known as **The Fundamental Theorem of Calculus**.

In fact, calculating definite integrals and evaluating indefinite integrals would be very difficult without this connection.

There are a number of forms of this theorem, but one version divides it into two parts:

For both parts we assume that *f* is continuous on some interval $[a,b]$, and $a \le x \le b$.

**Part 1**

Let $F(x)=\int_a^x f(t)\,dt$, then

$$\frac{d}{dx}F(x)=f(x)$$

This tells us that if $F(x)$ is a function that calculates the area under a function $f(x)$ between some constant *a* and *x*, then the derivative with respect to *x* of $F(x)$ is $f(x)$.

**Part 2**

If $\frac{d}{dx}F(x)=f(x)$ then $\int_a^b f(x)\,dx=F(b)-F(a)$

This tells us that if $F(x)$ is an anti-derivative of $f(x)$, then we can calculate the area under $f(x)$ on the interval $[a,b]$ using $F(x)$.

A note about combining the chain rule with the fundamental theorem
(Example 5, page 370)

$\frac{d}{dx}\int_a^{x^4}\sin(t)\,dt$ In this case we must be careful since our endpoint is a function of the independent variable. To calculate this properly we must invoke the chain rule as follows:

First set $u = x^4$

Then our integral becomes $\frac{d}{dx}\left[\int_a^u \sin(t)\,dt\right]\frac{du}{dx} = \sin(u)\frac{du}{dx} = \sin\left(x^4\right)\cdot 4x^3$

**Using the Fundamental Theorem**

Some Useful Formulas for finding anti-derivatives (**Indefinite Integrals**):

1) $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $x \neq -1$

2) $\int \frac{1}{x} dx = \ln|x| + C$

3) $\int e^x dx = e^x + C$

4) $\int \cos(x) dx = \sin(x) + C$

5) $\int \sin(x) dx = -\cos(x) + C$

Note the following important pattern:

$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$ This comes up a lot, eg.

$\int \frac{2x+1}{x^2+x+4} dx$

Since $\frac{d}{dx}\left(x^2+x+4\right) = 2x+1$

$\int \frac{2x+1}{x^2+x+4} dx = \ln\left|x^2+x+4\right| + C$

An enhancement on this formula

$$\int \frac{f^{'}(x)}{f(x)^{n}}dx = -\left(\frac{1}{n-1}\right)\frac{1}{f(x)^{n-1}} + C$$

**Example:**

$$\int \frac{x}{\left(x^{2}+1\right)^{2}}dx$$

We know that $\frac{d}{dx}\left(x^{2}+1\right) = 2x$

So we have $\int \frac{x}{\left(x^{2}+1\right)^{2}}dx = \frac{1}{2}\int \frac{2x}{\left(x^{2}+1\right)^{2}}dx = \frac{1}{2}\left(-\frac{1}{1}\right)\frac{1}{x^{2}+1} + C = \left(-\frac{1}{2}\right)\frac{1}{x^{2}+1}$

**Example**, find: $\int \sec(x)\tan(x)dx$

$$\int \sec(x)\tan(x)dx = \int \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}dx = \int \frac{\sin(x)}{\cos(x)^{2}}dx = -\int \frac{-\sin(x)}{\cos(x)^{2}}dx$$

But $\left[\cos(x)\right]^{'} = -\sin(x)$

So we have $\int \sec(x)\tan(x)dx = -\left(-\frac{1}{\cos(x)}\right) + C = \frac{1}{\cos(x)} + C = \sec(x) + C$

**Another Pattern**

Recall that $\frac{d}{dx}\sin^{-1}(x)=\frac{1}{\sqrt{1-x^2}}$

How do we know this?

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$$

If $x=\sin(y)$ then $\frac{dx}{dy}=\cos(y)$

$$\frac{dy}{dx}=\frac{1}{\cos(y)}=\frac{1}{\sqrt{1-\sin^2(y)}}=\frac{1}{\sqrt{1-x^2}}$$

So clearly, we have $\int \frac{1}{\sqrt{1-x^2}}\,dx=\sin^{-1}(x)+C$

But likewise, $\int \frac{f'(x)}{\sqrt{1-f(x)^2}}\,dx=\sin^{-1}(f(x))+C$

Example: $\int \frac{x}{\sqrt{1-x^4}}\,dx$

Since $\frac{d}{dx}x^2=2x$

$$\int \frac{x}{\sqrt{1-x^4}}\,dx=\frac{1}{2}\int \frac{2x}{\sqrt{1-\left(x^2\right)^2}}\,dx=\frac{1}{2}\sin^{-1}\left(x^2\right)+C$$

**A similarly useful formula**

$$\int \frac{f'(x)}{1+f(x)^2}\,dx = \tan^{-1}(x) + C$$

**Example of Simplifying before integrating**

$$\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2} dx = \int_1^9 2 + t^{1/2} - \frac{1}{t^2}\,dx = \left[2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\right]_1^9 =$$

$$18 + \frac{2}{3}\cdot 27 + \frac{1}{9} - \left(2 + \frac{2}{3} + 1\right) = 36 - 3 + \frac{1}{9} - \frac{2}{3} = 33 + \frac{1-6}{9} = 33 - \frac{5}{9} = 32\frac{4}{9}$$

Give Students Handout 3 with some examples to try:

**Net Change**

We can rewrite

$$\int_a^b f(x)dx = F(b) - F(a)$$

as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Where $F'(x)$ is the rate of change of $F(x)$

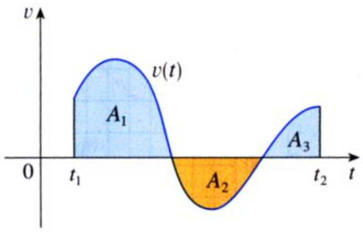
Example:

Consider a reservoir which has water flowing into or out of it at a rate of $V'(t)$

So $V(t_2) - V(t_1) = \int_{t_1}^{t_2} V'(t)dt$ is the change in the amount of water between time $t_1$ and $t_2$.

**Displacement vs. distance traveled.**

Consider a train that travels back and forth on a straight rail at a velocity according to this graph.




If you wish to know it's displacement then

displacement = $A_1 - A_2 + A_3 = \int_{t_1}^{t_2} v(t)\,dt$

If instead you want to know the distance traveled then you want

distance = $A_1 + A_2 + A_3 = \int_{t_1}^{t_2} |v(t)|\,dt$

Example from the book:

A particle has velocity $v(t)=t^2-t-6$

Find the displacement and distance traveled between 1 and 4 seconds:

displacement = $\int_1^4 \left(t^2-t-6\right)dt=$

$$\left[\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_1^4=\frac{64}{3}-8-24-\left(\frac{1}{3}-\frac{1}{2}-6\right)=-\frac{9}{2}$$

distance = $\int_1^4 \left|t^2-t-6\right|dt$

Find where the direction changes: $t^2-t-6=(t-3)(t+2)=0$

So it changes at -2 and 3 seconds.

$$\int_1^4 \left|t^2-t-6\right|dt=\int_1^3 \left|t^2-t-6\right|dt+\int_3^4 \left|t^2-t-6\right|dt=$$

$$\left|\left[\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_1^3\right|+\left|\left[\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_3^4\right|=$$

$$\left|9-\frac{9}{2}-18-\left(\frac{1}{3}-\frac{1}{2}-6\right)\right|+\left|\frac{64}{3}-8-24-\left(9-\frac{9}{2}-18\right)\right|=$$

$$\left|-\frac{22}{3}\right|+\left|\frac{17}{6}\right|=\frac{44}{6}+\frac{17}{6}=\frac{61}{6}$$