Math 109 Calc 1 Lecture 33

Substitution 5.5

Before we get to today's main subject, I'd like to review some indefinite integrals that we already have some experience with.

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

We can expand this a little

$$\int \frac{1}{x+a}\,dx = \ln|x+a| + C$$

This looks the same because when we use the chain rule on this function $\frac{d}{dx}x + a = 1$

One more enhancement

$$\int \frac{1}{bx+a}\,dx = \frac{1}{b}\int \frac{1}{x+a/b}\,dx = \frac{1}{b}\ln|x+a/b| + C$$

Now look at the following integral

$$\int \frac{x^2+3x+1}{x+1}\,dx$$

Unfortunately, we can't factor the numerator, so there is no chance on a nice simplification. However, we can just try long division and see what happens.

$$\begin{array}{r|l} & x+2 \\ \hline x+1 & x^2+3x+1 \\ & x^2+x \\ & \quad 2x+1 \\ & \quad 2x+2 \\ & \qquad -1 \end{array}$$

So, we have

$$\int \frac{x^2+3x+1}{x+1}\,dx = \int x+2-\frac{1}{x+1}\,dx = \frac{x^2}{2} + 2x + \ln|x+1| + C$$

This technique can come in handy.

Examples: to try in class

$$\int_1^2 \frac{x^4 - 4x}{x^3}\,dx$$

$$\int_{-1}^1 \frac{x}{\left(x^2+1\right)^4}\,dx$$

$$\int_x^{x^2} (1-t)^4\,dt$$

$$\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$

Substitution

We start with the chain rule:
$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ where $y = f(x)$ and $u = g(x)$

So, in Newtonian notation

$$\left[f\left(g(x)\right)\right]' = f'\left(g(x)\right) g'(x)$$

Note that $\frac{du}{dx} = \frac{d}{dx} g(x)$

As a matter of notation, we write this as

$$du = g'(x)\, dx$$

Keep in mind, that this is notation, not the multiplying of the denominator of a fraction.

If we integrate both sides of the equation, we get

$$\int \left[f\left(g(x)\right)\right]' dx = f\left(g(x)\right) = \int f'\left(g(x)\right) g'(x)\, dx = \int f'(u)\, du$$

This gives us the substitution formula for indefinite integrals

$$\int f'\left(g(x)\right) g'(x)\, dx = \int f'(u)\, du$$

This provides us with a technique for evaluation integrals in a more direct way than the reverse engineering we've been using.

Example:

$$\int \sqrt{2x+1}\, dx$$

$$u = 2x+1$$

$$\frac{du}{dx} = 2$$

We use a notational convention here, acting like *du* and *dx* are separate quantities, which they are not.

$$du = 2dx \rightarrow dx = \frac{1}{2}du$$

Substituting we get

$$\int \sqrt{2x+1}\, dx = \int \sqrt{u} \cdot \frac{1}{2} du = \frac{1}{3}u^{3/2} + C$$

The last step is to substitute back for *x*.

$$\frac{1}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

We can check this easily by finding the derivative

$$\frac{d}{dx}\frac{1}{3}(2x+1)^{3/2} + C = \frac{3}{2} \cdot \frac{1}{3}(2x+1)^{3/2-1} \cdot 2 + 0 = \sqrt{2x+1}$$

There is sometimes more than one substitution that will work.

With the intent of getting rid of the square root we let
$u^2 = 2x+1$

So, we have
$u = \sqrt{2x+1}$

Using the notational shorthand again, we get

$$du = \frac{1}{\sqrt{2x+1}}\,dx$$

or

$$du = \frac{1}{u}\,dx \rightarrow dx = u\ du$$

Now we can do the substitution as follows
$$\int \sqrt{2x+1}\ dx = \int \sqrt{u^2} \cdot u\ du = \int u \cdot u\ du = \int u^2 du = \frac{u^3}{3} + C$$

Now substituting back, we get

$$\frac{u^3}{3} = \frac{1}{3}\left(2x+1\right)^{3/2} + C$$

Note: Not every substitution will work. With some practice you will learn to guess correctly.

Example:

$$\int \frac{1}{1+\sqrt{x}}\,dx$$

To eliminate the square root, we let $x = u^2$ or $u = \sqrt{x}$

This gives us $dx = 2u\,du$

Substituting we get

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{1}{1+u}\cdot 2u\,du = \int \frac{2u}{1+u}\,du$$

Now we use the technique we saw at the beginning of class

$$\begin{array}{r|l} & \quad 2 \\ \hline u+1 & 2u \\ & 2u+2 \\ & \quad -2 \end{array}$$

So, we have

$$\int \frac{2u}{1+u}\,du = \int 2 - \frac{2}{1+u}\,du = 2u - 2\ln|1+u| + C$$

Now we substitute back

$$2u - 2\ln|1+u| + C = 2\sqrt{x} - 2\ln\left|1+\sqrt{x}\right| + C$$

Example:

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx$$

Here we let $u = 1 - 4x^2$

$$du = -8x\,dx \qquad x\,dx = -\frac{du}{8}$$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx = \int \frac{-1}{8\sqrt{u}}\,du = \int \frac{1}{\sqrt{u}}\left(-\frac{du}{8}\right) = -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{8}\left(2u^{1/2}\right) + C$$

Substituting back

$$-\frac{1}{8}\left(2u^{1/2}\right) + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

As an alternative:

$$x = \frac{1}{2}\sin u$$

$$dx = \frac{1}{2}\cos u\, du$$

Substituting

$$\int \frac{x}{\sqrt{1-4x^2}}\, dx = \int \frac{\sin u}{\sqrt{1-\sin^2 u}}\, \frac{1}{2}\cos u\ du$$

Since

$$1-\sin^2 u = \cos^2 u$$

$$\sqrt{1-\sin^2 u} = \cos u$$

$$\int \frac{\frac{1}{2}\sin u}{\sqrt{1-\sin^2 u}}\, \frac{1}{2}\cos u\ du = \frac{1}{4}\int \frac{\sin u}{\cos u}\cos u\ du = \frac{1}{4}\int \sin u\ du = -\frac{\cos u}{4} + C$$

Substituting back, since

$$x = \frac{1}{2}\sin u \rightarrow x^2 = \frac{\sin^2 u}{4} = \frac{1-\cos^2 u}{4}$$

$$\frac{1-\cos^2 u}{4} = x$$

$$1-\cos^2 u = 4x$$

$$\cos^2 u = 1-4x$$

$$\cos u = \sqrt{1-4x}$$

So $\int \frac{x}{\sqrt{1-4x^2}}\, dx = -\frac{1}{4}\sqrt{1-4x^2} + C$

Definite Integrals and Substitution

Recall the first Problem

$$\int \sqrt{2x+1}\,dx = \frac{1}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

We can evaluate the integral $\int_1^4 \sqrt{2x+1}\,dx$ in two ways

$$\left[\frac{1}{3}(2x+1)^{3/2}\right]_1^4 = \frac{1}{3}\left[9^{3/2} - 3^{3/2}\right] = 9 - \sqrt{3}$$

Or noting that when $x = 1$ $u = 3$ and when $x = 4$ $u = 9$

Plugging in $\frac{1}{3}\left[u^{3/2}\right]_3^9 = \frac{1}{3}\left[27 - 3\sqrt{3}\right] = 9 - \sqrt{3}$

This Substitution rule for definite integrals looks like this:

$$\int_a^b f'(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,dx$$

One more example:

$$\int \frac{\ln x}{x}\,dx$$

Substitute $u = \ln x$

$du = \frac{dx}{x}$

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{u^2}{2} = \frac{\ln^2 x}{2} + C$$

A quick check

$$\frac{d}{dx}\frac{\ln^2 x}{2} + C = \frac{2\ln x}{2} \cdot \frac{1}{x} + 0 = \frac{\ln x}{x}$$

Example:

Examples: to try in class

$$\int_0^4 \frac{x}{\sqrt{x+4}}\,dx$$

Try u = x+4

$$\int \frac{\cos(x)}{1+\sin^2(x)}\,dx$$